Math 302.102 Fall 2010
Summary of Discrete Random Variables

A random variable X is called *discrete* if it can assume at most countably many values. A discrete random variable can be completely described by its *probability mass function* (also called its *probability function* or *mass function* or *probability density function* or *density function*) which is just the values of

$$\mathbf{P}\{X = k\}$$

for $k \in \mathbb{Z}$.

Example. We say that a random variable X has a *Bernoulli distribution* with parameter p for some $0 \le p \le 1$ if

$$\mathbf{P}\{X = 1\} = p \quad \text{and} \quad \mathbf{P}\{X = 0\} = 1 - p.$$

Note that

$$\mathbb{E}(X) = p \quad \text{and} \quad \mathrm{Var}(X) = p(1-p).$$

Often we write $X \sim \text{Bernoulli}(p)$ for such a random variable.

Example. We say that a random variable X has a *binomial distribution* with parameters n and p for some positive integer n and some $0 \le p \le 1$ if

$$\mathbf{P}\{X = k\} = \binom{n}{k} p^k (1-p)^{n-k}$$

for $k = 0, 1, 2, \ldots, n$. Note that

$$\mathbb{E}(X) = np \quad \text{and} \quad \mathrm{Var}(X) = np(1-p).$$

Often we write $X \sim \text{Bin}(n, p)$ for such a random variable.

Example. We say that a random variable X has a *Poisson distribution* with parameter λ for some $\lambda > 0$ if

$$\mathbf{P}\{X = k\} = \frac{\lambda^k}{k!} e^{-\lambda}$$

for $k = 0, 1, 2, \ldots, n$. Note that

$$\mathbb{E}(X) = \lambda \quad \text{and} \quad \mathrm{Var}(X) = \lambda.$$

Often we write $X \sim \text{Poisson}(\lambda)$ for such a random variable.

Example. We say that a random variable X has a *geometric distribution* with parameter p for some $0 < p < 1$ if

$$\mathbf{P}\{X = k\} = (1-p)^{k-1} p$$

for $k = 1, 2, \ldots, n$. Note that

$$\mathbb{E}(X) = \frac{1}{p} \quad \text{and} \quad \mathrm{Var}(X) = \frac{1-p}{p^2}.$$

Often we write $X \sim \text{Geometric}(p)$ for such a random variable.

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Calculations of Means and Variances for Discrete Random Variables

If X is a discrete random variable, then its expected value $\mathbb{E}(X)$ is just the weighted average of its possible values; that is,

$$E(X) = \sum_{k=-\infty}^{\infty} k\mathbf{P}\left\{X = k\right\}.$$

Its second moment $\mathbb{E}(X^2)$ is just the weighted average of its possible values squared; that is,

$$E(X^2) = \sum_{k=-\infty}^{\infty} k^2\mathbf{P}\left\{X = k\right\}.$$

Furthermore, $\mathrm{Var}(X)$ can be computed as $\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$.

Example. If $X \sim \text{Bernoulli}(p)$, then

$$E(X) = 0 \cdot \mathbf{P}\left\{X = 0\right\} + 1 \cdot \mathbf{P}\left\{X = 1\right\} = 0 \cdot (1-p) + 1 \cdot p = p$$

and

$$E(X^2) = 0^2 \cdot \mathbf{P}\left\{X = 0\right\} + 1^2 \cdot \mathbf{P}\left\{X = 1\right\} = 0^2 \cdot (1-p) + 1^2 \cdot p = p.$$

so that

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = p - p^2 = p(1-p).$$

Example. If $X \sim \text{Bin}(n,p)$, then we can express X as $X = X_1 + X_2 + \cdots + X_n$ where $X_1, \ldots, X_n$ are iid $\text{Bernoulli}(p)$ random variables. Thus,

$$\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_n) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_n) = p + \cdots + p = np$$

and

$$\mathrm{Var}(X) = \mathrm{Var}(X_1 + \cdots + X_n) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n) = p(1-p) + \cdots + p(1-p) = np(1-p).$$

In order to calculate the variance of both a geometric random variable and a Poisson random variable, it turns out that it is not so easy to calculate the second moment directly. Instead, it turns out to be easier to consider $\mathbb{E}[X(X-1)]$. This is not a problem since

$$\mathbb{E}[X(X-1)] = \mathbb{E}(X^2 - X) = \mathbb{E}(X^2) - \mathbb{E}(X).$$

Thus, if we can calculate $\mathbb{E}[X(X-1)]$, then we can find $\mathbb{E}(X^2) = \mathbb{E}[X(X-1)] + \mathbb{E}(X)$.

Example. If $X \sim \text{Poisson}(\lambda)$, then

$$\mathbb{E}(X) = \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k}{k!}e^{-\lambda} = e^{-\lambda}\sum_{k=1}^{\infty} k \cdot \frac{\lambda^k}{k!} = \lambda e^{-\lambda}\sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda}\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = \lambda e^{-\lambda}e^{\lambda} = \lambda$$

and

$$\mathbb{E}[X(X-1)] = \sum_{k=0}^{\infty} k(k-1)\cdot\frac{\lambda^k}{k!}e^{-\lambda} = e^{-\lambda}\sum_{k=2}^{\infty} k(k-1)\cdot\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=2}^{\infty}\frac{\lambda^k}{(k-2)!}$$
$$= \lambda^2 e^{-\lambda}\sum_{k=2}^{\infty}\frac{\lambda^{k-2}}{(k-2)!}$$
$$= \lambda^2 e^{-\lambda}\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}$$
$$= \lambda^2 e^{-\lambda}e^{\lambda}$$
$$= \lambda^2$$

so that

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \mathbb{E}[X(X-1)] + \mathbb{E}(X) - [\mathbb{E}(X)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

Example. If $X \sim \text{Geometric}(p)$, then in order to compute $\mathbb{E}(X)$ and $\mathrm{Var}(X)$, we need to sum up some particular series. To begin, note that if $0 < q < 1$, then

$$\sum_{k=1}^{\infty} kq^{k-1} = \sum_{k=1}^{\infty}\frac{\mathrm{d}}{\mathrm{d}q}q^k = \frac{\mathrm{d}}{\mathrm{d}q}\sum_{k=1}^{\infty}q^k = \frac{\mathrm{d}}{\mathrm{d}q}\left[\left(\sum_{k=0}^{\infty}q^k\right)-1\right] = \frac{\mathrm{d}}{\mathrm{d}q}\left[\frac{1}{1-q}-1\right] = \frac{1}{(1-q)^2}.$$

Furthermore,

$$\sum_{k=2}^{\infty} k(k-1)q^{k-2} = \sum_{k=2}^{\infty}\frac{\mathrm{d}^2}{\mathrm{d}q^2}q^k = \frac{\mathrm{d}^2}{\mathrm{d}q^2}\sum_{k=2}^{\infty}q^k = \frac{\mathrm{d}^2}{\mathrm{d}q^2}\left[\left(\sum_{k=0}^{\infty}q^k\right)-1-q\right] = \frac{\mathrm{d}^2}{\mathrm{d}q^2}\left[\frac{1}{1-q}-1-q\right]$$
$$= \frac{2}{(1-q)^3}.$$

Thus, we see that

$$\mathbb{E}(X) = \sum_{k=1}^{\infty} k\cdot(1-p)^{k-1}p = p\sum_{k=1}^{\infty}k(1-p)^{k-1} = \frac{p}{(1-(1-p))^2} = \frac{p}{p^2} = \frac{1}{p}$$

and

$$\mathbb{E}[X(X-1)] = \sum_{k=1}^{\infty} k(k-1)\cdot(1-p)^{k-1}p = p\sum_{k=2}^{\infty}k(k-1)(1-p)^{k-1} = p(1-p)\sum_{k=2}^{\infty}k(k-1)(1-p)^{k-2}$$
$$= \frac{2p(1-p)}{(1-(1-p))^3}$$
$$= \frac{2p(1-p)}{p^3}$$
$$= \frac{2(1-p)}{p^2}$$

so that

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \mathbb{E}[X(X-1)] + \mathbb{E}(X) - [\mathbb{E}(X)]^2 = \frac{2(1-p)}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{1-p}{p^2}.$$