Math 302.102 Fall 2010
The Poisson Process

On October 29, 2010, we discussed the following exercise in class.

**Exercise.** Based on studies from previous Hallowe'ens, I have determined that trick-or-treating ghouls arrive at my door as follows. The ghouls arrive independently and the time in minutes between the arrival of successive ghouls is exponentially distributed with parameter $\lambda = 2$. If I leave my jack-o-latern lit for two hours (signifying that the ghouls are welcome to ring my bell) and my pumpkin contains 100 candy bars, do I expect to have enough candy bars to give one to every ghoul who trick-or-treats?

Our intuition told us that since the times in minutes between the arrivals of successive ghouls were independent and exponentially distributed with mean $1/\lambda = 1/2$, we should expect a ghoul every 30 seconds. Hence, in two hours we should expect 240 ghouls meaning that our box of 100 candy bars is not sufficient.

Let's prove that this intuition is correct. In fact, we can prove things in a bit more generality. Instead of assuming that the time in minutes between the arrival of successive ghouls is exponentially distributed with parameter $\lambda = 2$, let's assume that it is exponentially distributed with parameter $\lambda > 0$.

We now need to introduce some notation. Set $T_0 = 0$, and let $T_k$, $k = 1, 2, 3, \ldots$, denote the time that the $k$th ghoul arrives. Note that the arrival times are *random variables.* We have assumed that the times in minutes *between* the arrivals are independent and exponentially distributed. That is, if

$$W_k = T_k - T_{k-1}$$

then $W_k$ represents the waiting time between the arrival of ghoul $k-1$ and ghoul $k$. Furthermore, $W_1, W_2, \ldots$ are independent and identically distributed with $W_k \sim \text{Exp}(\lambda)$. Also note that

$$\begin{aligned}
T_k &= (T_0 - T_0) + (T_1 - T_1) + (T_2 - T_2) + \cdots + (T_{k-1} - T_{k-1}) + T_k \\
&= (T_1 - T_0) + (T_2 - T_1) + \cdots + (T_k - T_{k-1}) \\
&= W_1 + W_2 + \cdots + W_k.
\end{aligned}$$

In other words, $T_k$ is the sum of $k$ iid random variables each having an $\text{Exp}(\lambda)$ distribution. To find the distribution of $T_k$, we can use moment generating functions. That is,

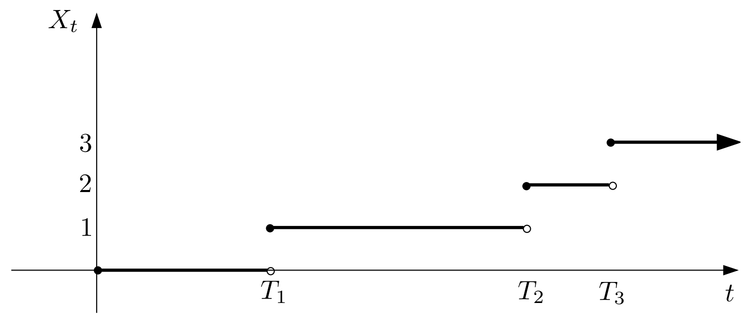
$$\begin{aligned}
m_{T_k}(t) = \mathbb{E}[e^{tT_k}] = \mathbb{E}[e^{t(W_1+\cdots+W_k)}] = \mathbb{E}[e^{tW_1}\cdots e^{tW_k}] = \mathbb{E}[e^{tW_1}]\cdots\mathbb{E}[e^{tW_k}] &= \left(\mathbb{E}[e^{tW_1}]\right)^k \\
&= \left(\frac{\lambda}{\lambda - t}\right)^k
\end{aligned}$$

which just so happens to be the moment generating function of a random variable with a $\text{Gamma}(k, \lambda)$ distribution. Thus,

$$T_k \sim \text{Gamma}(k, \lambda)$$

so that

$$\mathbf{P}\{T_k > t\} = \int_t^\infty \frac{\lambda^k}{\Gamma(k)} x^{k-1} e^{-\lambda x}\, \mathrm{d}x. \qquad (*)$$




Now, here is the next observation. Let $X_t$ denote the number of ghouls that have arrived by time $t$. Note that $X_t$ is a discrete random variable since the number of ghouls who have arrived is necessarily a non-negative integer. Observe that at most $k$ ghouls have arrived at time $t$ if and only if ghoul $k+1$ arrived after time $t$; that is, $\{X_t \le k\} = \{T_{k+1} > t\}$ and so

$$\mathbf{P}\{X_t \le k\} = \mathbf{P}\{T_{k+1} > t\}. \tag{$**$}$$

Our goal is to compute

$$\mathbb{E}(X_t) = \sum_{k=0}^{\infty} k\mathbf{P}\{X_t = k\} = \sum_{k=1}^{\infty} k\mathbf{P}\{X_t = k\}$$

but we do not yet have an expression for $\mathbf{P}\{X_t = k\}$. However, we do have an expression for $\mathbf{P}\{X_t \le k\}$. Observe that

$$\mathbf{P}\{X_t = k\} = \mathbf{P}\{X_t \le k\} - \mathbf{P}\{X_t \le k-1\}$$

and so using $(*)$ and $(**)$ we conclude

$$\begin{aligned}\mathbf{P}\{X_t = k\} &= \mathbf{P}\{T_{k+1} > t\} - \mathbf{P}\{T_k > t\} \\ &= \int_t^{\infty} \frac{\lambda^{k+1}}{\Gamma(k+1)} x^k e^{-\lambda x}\,\mathrm{d}x - \int_t^{\infty} \frac{\lambda^k}{\Gamma(k)} x^{k-1} e^{-\lambda x}\,\mathrm{d}x.\end{aligned} \tag{$\dagger$}$$

The next step is to analyze the difference of integrals of gamma functions. The "usual" way to analyze integrals of gamma function is to try integration by parts. Hence, consider

$$\int_t^{\infty} \lambda x^k e^{-\lambda x}\,\mathrm{d}x.$$

If we let $u = x^k$ and $\mathrm{d}v = \lambda e^{-\lambda x}\,\mathrm{d}x$, then

$$\int_t^{\infty} \lambda x^k e^{-\lambda x}\,\mathrm{d}x = x^k e^{-\lambda x}\Big|_t^{\infty} + \int_t^{\infty} k x^{k-1} e^{-\lambda x}\,\mathrm{d}x = t^k e^{-\lambda t} + k\int_t^{\infty} x^{k-1} e^{-\lambda x}\,\mathrm{d}x.$$

Multiplying both sides by $\lambda^k/\Gamma(k+1)$ implies

$$\frac{\lambda^k}{\Gamma(k+1)} \int_t^{\infty} \lambda x^k e^{-\lambda x}\,\mathrm{d}x = \frac{\lambda^k}{\Gamma(k+1)} t^k e^{-\lambda t} + \frac{\lambda^k}{\Gamma(k+1)} k \int_t^{\infty} x^{k-1} e^{-\lambda x}\,\mathrm{d}x$$

and so
$$\int_t^\infty \frac{\lambda^{k+1}}{\Gamma(k+1)} x^k e^{-\lambda x}\,\mathrm{d}x = \frac{(\lambda t)^k}{\Gamma(k+1)} e^{-\lambda t} + \int_t^\infty \frac{\lambda^k}{\Gamma(k)} x^{k-1} e^{-\lambda x}\,\mathrm{d}x$$
using the fact that $\Gamma(k+1) = k\Gamma(k)$ which in turn implies that

$$\int_t^\infty \frac{\lambda^{k+1}}{\Gamma(k+1)} x^k e^{-\lambda x}\,\mathrm{d}x - \int_t^\infty \frac{\lambda^k}{\Gamma(k)} x^{k-1} e^{-\lambda x}\,\mathrm{d}x = \frac{(\lambda t)^k}{\Gamma(k+1)} e^{-\lambda t}.$$

In other words, we have found our required expression for the particular difference of integrals of gamma functions in (†). Thus,

$$\mathbf{P}\left\{X_t = k\right\} = \frac{(\lambda t)^k}{\Gamma(k+1)} e^{-\lambda t} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

since $\Gamma(k+1) = k!$ when $k$ is a positive integer.

Finally, we conclude

$$\begin{aligned}
\mathbb{E}(X_t) = \sum_{k=1}^\infty k\mathbf{P}\left\{X_t = k\right\} = \sum_{k=1}^\infty k\frac{(\lambda t)^k}{k!} e^{-\lambda t} = e^{-\lambda t} \sum_{k=1}^\infty \frac{(\lambda t)^k}{(k-1)!} &= e^{-\lambda t} \sum_{k=1}^\infty \frac{(\lambda t)(\lambda t)^{k-1}}{(k-1)!} \\
&= \lambda t e^{-\lambda t} \sum_{k=1}^\infty \frac{(\lambda t)^{k-1}}{(k-1)!} \\
&= \lambda t e^{-\lambda t} \sum_{k=0}^\infty \frac{(\lambda t)^k}{k!} \\
&= \lambda t e^{-\lambda t} e^{\lambda t} \\
&= \lambda t.
\end{aligned}$$

Thus, if we return to the ghoulish exercise that started this whole mess and take $t = 120$ minutes and $\lambda = 2$, then
$$\mathbb{E}(X_{120}) = 2 \cdot 120 = 240$$
as suspected!