Math 302.102 Fall 2010
The Maximum and Minimum of Two IID Random Variables

Suppose that X_1 and X_2 are independent and identically distributed (iid) continuous random variables. By *independent*, we mean that

$$\mathbf{P}\{X_1 \in A, X_2 \in B\} = \mathbf{P}\{X_1 \in A\}\,\mathbf{P}\{X_2 \in B\}$$

for any $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$. By *identically distributed* we mean that X_1 and X_2 each have the same distribution function F (and therefore the same density function f).

Two quantities of interest are the maximum and minimum of X_1 and X_2. It turns out to be surprisingly easy to determine the distribution and density functions of the maximum and minimum.

The two key observations are that

$$\max\{X_1, X_2\} \le x \text{ if and only if both } X_1 \le x \text{ and } X_2 \le x$$

and

$$\min\{X_1, X_2\} > x \text{ if and only if both } X_1 > x \text{ and } X_2 > x.$$

In other words, an upper bound for the maximum gives an upper bound for each of X_1 and X_2, while a lower bound for the minimum gives a lower bound for each of X_1 and X_2.

Distribution of $\max\{X_1, X_2\}$

Suppose that $X = \max\{X_1, X_2\}$. By definition, the distribution function of X is

$$\begin{aligned} F_X(x) = \mathbf{P}\{X \le x\} = \mathbf{P}\{\max\{X_1, X_2\} \le x\} &= \mathbf{P}\{X_1 \le x \text{ and } X_2 \le x\} \\ &= \mathbf{P}\{X_1 \le x\,,\ X_2 \le x\} \\ &= \mathbf{P}\{X_1 \le x\}\,\mathbf{P}\{X_2 \le x\} \end{aligned}$$

using the fact that X_1 and X_2 are independent. However, both X_1 and X_2 have the *same* distribution function F and the *same* density function f. This means that

$$\mathbf{P}\{X_1 \le x\} = F(x) = \int_{-\infty}^{x} f(x)\,\mathrm{d}x \quad \text{and} \quad \mathbf{P}\{X_2 \le x\} = F(x) = \int_{-\infty}^{x} f(x)\,\mathrm{d}x.$$

Therefore,

$$F_X(x) = F(x) \cdot F(x) = [F(x)]^2.$$

The density function of $X = \max\{X_1, X_2\}$ can now be found by differentiation, namely

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x) = \frac{\mathrm{d}}{\mathrm{d}x}[F(x)]^2 = 2F(x)F'(x) = 2f(x)F(x).$$

Distribution of $\min\{X_1, X_2\}$

Suppose that $Y = \min\{X_1, X_2\}$. By definition, the distribution function of Y is

$$F_Y(y) = \mathbf{P}\left\{Y \le y\right\} = \mathbf{P}\left\{\min\{X_1, X_2\} \le y\right\}.$$

However, knowing an upper bound on the minimum is not of any use to us. Instead, we consider

$$F_Y(y) = \mathbf{P}\left\{Y \le y\right\} = 1 - \mathbf{P}\left\{Y > y\right\} = 1 - \mathbf{P}\left\{\min\{X_1, X_2\} > y\right\}.$$

This *is* useful to us since

$$\mathbf{P}\left\{\min\{X_1, X_2\} > y\right\} = \mathbf{P}\left\{X_1 > y\,,\, X_2 > y\right\} = \mathbf{P}\left\{X_1 > y\right\}\mathbf{P}\left\{X_2 > y\right\}$$

using the fact that X_1 and X_2 are independent. However, both X_1 and X_2 have the *same* distribution function F and the *same* density function f. This means that

$$\mathbf{P}\left\{X_1 > y\right\} = 1 - F(y) = \int_y^\infty f(x)\,\mathrm{d}x \quad \text{and} \quad \mathbf{P}\left\{X_2 > y\right\} = 1 - F(y) = \int_y^\infty f(x)\,\mathrm{d}x.$$

Therefore,

$$F_Y(y) = 1 - \mathbf{P}\left\{Y > y\right\} = 1 - [1 - F(y)] \cdot [1 - F(y)] = 1 - [1 - F(y)]^2.$$

The density function of $Y = \min\{X_1, X_2\}$ can now be found by differentiation, namely

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y}F_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y}\left(1 - [1 - F(y)]^2\right) = 2[1 - F(y)]F'(y) = 2f(y)[1 - F(y)].$$

Example. Suppose that X_1 and X_2 are independent random variables each having the $\text{Exp}(\lambda)$ distribution. Determine the density functions of $\max\{X_1, X_2\}$, and $\min\{X_1, X_2\}$.

Solution. Since X_1 and X_2 are iid $\text{Exp}(\lambda)$ random variables, they have common density function $f(x) = \lambda e^{-\lambda x}$, $x \ge 0$, and common distribution functions

$$F(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-\lambda x}, & x \ge 0. \end{cases}$$

Thus, the density function of $X = \max\{X_1, X_2\}$ is

$$f_X(x) = 2\lambda e^{-\lambda x}[1 - e^{-\lambda x}]$$

for $x \ge 0$, and the density function of $Y = \min\{X_1, X_2\}$ is

$$f_Y(y) = 2\lambda e^{-2\lambda y}$$

for $y \ge 0$. Note that we recognize the distribution of Y as $Y \sim \text{Exp}(2\lambda)$.