Math 302.102 Fall 2010
Summary of Lecture from September 29, 2010

Until now, all of our examples have involved sample spaces with a finite number of outcomes. We would now like to consider the case where the sample space contains a continuum of outcomes; that is, we want to have the sample space $S = \mathbb{R}$. One way to define probabilities for subsets of $\mathbb{R}$ is through the use of an auxiliary function known as a *probability density function*. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ has the properties that

(a) $f(x) \ge 0$ for all $x \in \mathbb{R}$, and

(b) $\displaystyle\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$.

If we define

$$\mathbf{P}\{A\} = \int_A f(x)\,\mathrm{d}x$$

for any event $A \subset \mathbb{R}$, then this defines a probability. Note that

$$\mathbf{P}\{\emptyset\} = \int_{\emptyset} f(x)\,\mathrm{d}x = 0, \quad \mathbf{P}\{S\} = \mathbf{P}\{\mathbb{R}\} = \int_{\mathbb{R}} f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1,$$

and if A and B are disjoint, then

$$\mathbf{P}\{A \cup B\} = \int_{A\cup B} f(x)\,\mathrm{d}x = \int_A f(x)\,\mathrm{d}x + \int_B f(x)\,\mathrm{d}x = \mathbf{P}\{A\} + \mathbf{P}\{B\}.$$

Furthermore, since $f \ge 0$, we conclude that

$$\mathbf{P}\{A\} = \int_A f(x)\,\mathrm{d}x \ge 0$$

and that since $A \subseteq \mathbb{R}$, we have

$$\mathbf{P}\{A\} = \int_A f(x)\,\mathrm{d}x \le \int_{\mathbb{R}} f(x)\,\mathrm{d}x = 1.$$

That is, $0 \le \mathbf{P}\{A\} \le 1$ so that $\mathbf{P}$ is a legitimate probability.

The following six examples of density functions are of particular importance for this course. We will be using them continually.

Example 1. Suppose that $\lambda > 0$ and let

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0, \\ 0, & x < 0. \end{cases}$$

Example 2. Suppose that $-\infty < a < b < \infty$ and let

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b, \\ 0, & \text{otherwise.} \end{cases}$$

Example 3. Suppose that $-\infty < \theta < \infty$ and let

$$f(x) = \frac{1}{\pi} \cdot \frac{1}{1 + (x - \theta)^2}$$

for $-\infty < x < \infty$.

Example 4. Suppose that $\lambda > 0$, $\alpha > 0$, and let

$$f(x) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Here, Γ is the gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, \mathrm{d}x.$$

Example 5. Suppose that $-\infty < \mu < \infty$, $\sigma > 0$, and let

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\}$$

for $-\infty < x < \infty$.

Example 6. Suppose that $a > 0$, $b > 0$, and let

$$f(x) = \begin{cases} \dfrac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

In order to verify that these six functions are legitimate density functions, we need to verify that each is non-negative and integrates to 1. Clearly all six are non-negative. As for the fact that each integrates to 1, the first three can be verified by direct integration.

Example 1.

$$\int_{-\infty}^{\infty} f(x) \, \mathrm{d}x = \int_{-\infty}^{0} f(x) \, \mathrm{d}x + \int_{0}^{\infty} f(x) \, \mathrm{d}x = \int_{-\infty}^{0} 0 \, \mathrm{d}x + \int_{0}^{\infty} \lambda e^{-\lambda x} \, \mathrm{d}x = -e^{-\lambda x} \Big|_0^\infty = 1.$$

Example 2.

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) \, \mathrm{d}x = \int_{-\infty}^{a} f(x) \, \mathrm{d}x + \int_{a}^{b} f(x) \, \mathrm{d}x + \int_{b}^{\infty} f(x) \, \mathrm{d}x &= \int_{-\infty}^{a} 0 \, \mathrm{d}x + \int_{a}^{b} \frac{\mathrm{d}x}{b-a} + \int_{b}^{\infty} 0 \, \mathrm{d}x \\ &= \frac{x}{b-a} \Big|_a^b = 1. \end{aligned}$$

Example 3.

$$\int_{-\infty}^{\infty} f(x) \, \mathrm{d}x = \int_{-\infty}^{\infty} \frac{1}{\pi} \cdot \frac{1}{1+(x-\theta)^2} \, \mathrm{d}x = \frac{1}{\pi} \arctan(x - \theta) \Big|_{-\infty}^{\infty} = \frac{1}{\pi} \left[\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right] = 1.$$

As for the remaining three examples, it will take some work to verify each of those actually integrates to 1.