Math 302.102 Fall 2010
A Confidence Interval for the Mean of a Normal Population with Known Variance

The purpose of this handout is to guide you through a rigorous derivation of the formula for a *confidence interval* for the mean μ from a normally distributed population with a known variance σ^2. The formula that appears in the very last part of this handout is always stated without any justification in elementary statistics class. Now you can justify it!

Throughout this handout, suppose that $X \sim \mathcal{N}(0,1)$ so that the density function of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

for $-\infty < x < \infty$.

Problem 1. Compute $\mathbb{E}(X) = \int_{-\infty}^{\infty} x f_X(x)\,\mathrm{d}x$. *Hint*: use symmetry.

Problem 2. Compute $\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,\mathrm{d}x$. *Hint*: use integration-by-parts with $u = x$ and $\mathrm{d}v = xe^{-x^2/2}\,\mathrm{d}x$, and the fact that a normal density function integrates to 1.

Problem 3. Compute $\mathrm{Var}(X)$.

Problem 4. Compute $m_X(t) = \mathbb{E}[e^{tX}]$, the moment generating function of X. *Hint*: complete the square in the exponent, manipulate it a bit, and use the fact that a normal density function integrates to 1.

For the next three problems, suppose that $Y = \sigma X + \mu$ where $\sigma > 0$ and $\mu \in \mathbb{R}$ are constant.

Problem 5. Compute $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$.

Problem 6. Determine the density function of Y and conclude that $Y \sim \mathcal{N}(\mu, \sigma^2)$. Note that this justifies the statement "Y is normally distributed with mean μ and variance σ^2 if and only if $Y \sim \mathcal{N}(\mu, \sigma^2)$."

Problem 7. Verify that the moment generating function of Y is $m_Y(t) = e^{\mu t + \sigma^2 t^2/2}$.

Problem 8. As a converse to the previous three problems, suppose that $Y \sim \mathcal{N}(\mu, \sigma^2)$ and let

$$Z = \frac{Y - \mu}{\sigma}.$$

Verify that $Z \sim \mathcal{N}(0,1)$.

Problem 9. Suppose that $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ are independent random variables. Let $Y = X_1 + X_2$ and determine the moment generating function of Y. As a consequence of Problem 7, this proves that $X_1 + X_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Problem 10. As an extension of the previous problem, show that if $X_1, X_2, \ldots, X_n$ are independent with $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, then

$$X_1 + X_2 + \cdots + X_n \sim \mathcal{N}(\mu_1 + \mu_2 + \cdots + \mu_n, \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2).$$

In particular, if $X_1, X_2, \ldots, X_n$ are iid with $X_i \sim \mathcal{N}(\mu, \sigma^2)$, then

$$X_1 + X_2 + \cdots + X_n \sim \mathcal{N}(n\mu, n\sigma^2).$$

Problem 11. Suppose that $X_1, X_2, \ldots, X_n$ are iid with $X_i \sim \mathcal{N}(\mu, \sigma^2)$, and let

$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$

Verify that $\overline{X} \sim \mathcal{N}(\mu, \sigma^2/n)$ and therefore conclude from Problem 8 that

$$\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0,1).$$

At this point we have all of the results needed to derive the formula for a confidence interval. Suppose that $Z \sim \mathcal{N}(0,1)$ and assume that $z_{\alpha/2}$ is chosen so that

$$\mathbf{P}\left\{-z_{\alpha/2} \le Z \le z_{\alpha/2}\right\} = 1 - \alpha.$$

For instance, from a table of normal probabilities we see that $z_{0.025} = 1.96$ since

$$\mathbf{P}\left\{-1.96 \le Z \le 1.96\right\} = 0.95$$

and $z_{0.05} = 1.645$ since

$$\mathbf{P}\left\{-1.645 \le Z \le 1.645\right\} = 0.90.$$

Hence, since

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0,1)$$

we conclude that

$$\mathbf{P}\left\{-z_{\alpha/2} \le \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \le z_{\alpha/2}\right\} = 1 - \alpha.$$

Solving for μ implies that

$$\mathbf{P}\left\{\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right\} = 1 - \alpha.$$

As a result of this probabilistic statement, we say that

$$\left[\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$$

is a $100(1-\alpha)\%$ confidence interval for μ (or a confidence interval for μ with coverage probability $1-\alpha$).

Solutions

Problem 1. Observe that

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f_X(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,\mathrm{d}x = 0$$

since the integrand is odd.

Problem 2. Since $\mathbb{E}(X) = 0$ from the previous problem, we see that

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,\mathrm{d}x.$$

To evaluate

$$\int_{-\infty}^{\infty} x^2 e^{-x^2/2}\,\mathrm{d}x$$

use integration-by-parts with $u = x$ and $\mathrm{d}v = xe^{-x^2/2}\,\mathrm{d}x$ so that

$$\int_{-\infty}^{\infty} x^2 e^{-x^2/2}\,\mathrm{d}x = -xe^{-x^2}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2/2}\,\mathrm{d}x = \int_{-\infty}^{\infty} e^{-x^2/2}\,\mathrm{d}x.$$

If we multiply by $1/\sqrt{2\pi}$, then using the fact that the density function of a $\mathcal{N}(0,1)$ random variable integrates to 1, we conclude

$$\mathrm{Var}(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\,\mathrm{d}x = 1.$$

Problem 3. We find $\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = 1 - 0^2 = 1$.

Problem 4. By definition,

$$m_X(t) = \mathbb{E}(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx - x^2/2}\,\mathrm{d}x.$$

Now, we complete the square in the exponent; that is, we write

$$tx - \frac{x^2}{2} = -\frac{1}{2}(x^2 - 2tx) = -\frac{1}{2}(x^2 - 2tx + t^2 - t^2) = -\frac{1}{2}(x^2 - 2tx + t^2) + \frac{t^2}{2} = -\frac{(x-t)^2}{2} + \frac{t^2}{2}$$

so that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx - x^2/2}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-t)^2}{2} + \frac{t^2}{2}}\,\mathrm{d}x = e^{t^2/2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-t)^2/2}\,\mathrm{d}x.$$

However, if we substitute $u = x - t$, then

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-t)^2}{2}}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2}\,\mathrm{d}u = 1$$

(since it is the integral of the density function of a $\mathcal{N}(0,1)$ random variable), and so we conclude

$$m_X(t) = e^{t^2/2}.$$

Problem 5. If $Y = \sigma X + \mu$, then

$$\mathbb{E}(Y) = \mathbb{E}(\sigma X + \mu) = \sigma \mathbb{E}(X) + \mu = 0 + \mu = \mu$$

and

$$\operatorname{Var}(Y) = \operatorname{Var}(\sigma X + \mu) = \operatorname{Var}(\sigma X) = \sigma^2 \operatorname{Var}(X) = \sigma^2.$$

Problem 6. If $Y = \sigma X + \mu$, then for any $y \in \mathbb{R}$, the distribution function of Y is

$$F_Y(y) = \mathbf{P}\{Y \le y\} = \mathbf{P}\{\sigma X + \mu \le y\} = \mathbf{P}\left\{X \le \frac{y-\mu}{\sigma}\right\} = \int_{-\infty}^{\frac{y-\mu}{\sigma}} f_X(x)\,\mathrm{d}x$$

so that

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right) \cdot \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{y-\mu}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\}.$$

Problem 7. If $Y = \sigma X + \mu$, then

$$m_Y(t) = \mathbb{E}[e^{tY}] = \mathbb{E}[e^{t(\sigma X + \mu)}] = \mathbb{E}[e^{t\mu} e^{\sigma t X}] = e^{t\mu}\mathbb{E}[e^{\sigma t X}] = e^{t\mu} m_X(\sigma t) = e^{t\mu} e^{(\sigma t)^2/2} = e^{t\mu + \sigma^2 t^2/2}.$$

Problem 8. Since $Y \sim \mathcal{N}(\mu, \sigma^2)$ and

$$Z = \frac{Y - \mu}{\sigma} = \frac{1}{\sigma} Y - \frac{\mu}{\sigma},$$

then the moment generating function of Z is

$$m_Z(t) = E[e^{tZ}] = \mathbb{E}\left[e^{t\left(\frac{1}{\sigma}Y - \frac{\mu}{\sigma}\right)}\right] = e^{-t\mu/\sigma} m_Y(t/\sigma) = e^{-t\mu/\sigma} e^{t\mu/\sigma + \sigma^2 t^2/(2\sigma^2)} = e^{t^2/2}$$

which is the moment generating function of a $\mathcal{N}(0,1)$ random variable. Thus, $Z \sim \mathcal{N}(0,1)$.

Problem 9. If $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, then from Problem 7 we know

$$m_{X_i}(t) = \mathbb{E}(e^{tX_i}) = e^{t\mu_i + \sigma_i^2 t^2/2}.$$

Therefore, the mgf of $X_1 + X_2$ is

$$\begin{aligned} m_{X_1+X_2}(t) = \mathbb{E}[e^{t(X_1+X_2)}] = \mathbb{E}[e^{tX_1} e^{tX_2}] = \mathbb{E}(e^{tX_1})\mathbb{E}(e^{tX_2}) &= e^{t\mu_1 + \sigma_1^2 t^2/2} e^{t\mu_1 + \sigma_1^2 t^2/2} \\ &= e^{t(\mu_1+\mu_2) + (\sigma_1^2 + \sigma_2^2)t^2/2} \end{aligned}$$

which we recognize as the mgf of a $\mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ random variable.

Problem 10. The mgf of $X_1 + \cdots + X_n$ is

$$\begin{aligned} m_{X_1+\cdots+X_n}(t) = \mathbb{E}[e^{t(X_1+\cdots+X_n)}] = \mathbb{E}[e^{tX_1} \cdots e^{tX_n}] &= \mathbb{E}(e^{tX_1}) \cdots \mathbb{E}(e^{tX_n}) \\ &= e^{t\mu_1 + \sigma_1^2 t^2/2} \cdots e^{t\mu_n + \sigma_n^2 t^2/2} \\ &= e^{t(\mu_1 + \cdots + \mu_n) + (\sigma_1^2 + \cdots + \sigma_n^2)t^2/2} \end{aligned}$$

which we recognize as the mgf of a $\mathcal{N}(\mu_1 + \cdots + \mu_n, \sigma_1^2 + \cdots + \sigma_n^2)$ random variable.

Problem 11. We know from the previous problem that if $X_1, \dots, X_n$ are iid $\mathcal{N}(\mu, \sigma^2)$, then the mgf of $X_1 + \cdots + X_n$ is

$$m_{X_1+\cdots+X_n}(t) = e^{n\mu t + n\sigma^2 t^2/2}.$$

Therefore, the mgf of $\overline{X}$ is

$$m_{\overline{X}}(t) = \mathbb{E}(e^{t\overline{X}}) = \mathbb{E}[e^{\frac{t}{n}(X_1+\cdots+X_n)}] = m_{X_1+\cdots+X_n}(t/n) = e^{n\mu(t/n)+n\sigma^2(t/n)^2/2} = e^{\mu t+\sigma^2 t^2/(2n)}$$

which we recognize as the mgf of a $\mathcal{N}(\mu, \sigma^2/n)$ random variable. We can now conclude from Problem 8 that

$$\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0, 1).$$