Math 302.102 Fall 2010
The Gamma Function

Suppose that $p > 0$, and define

$$\Gamma(p) = \int_0^\infty u^{p-1}\, e^{-u}\, du.$$

We call $\Gamma(p)$ the *Gamma function* and it appears in many of the formulas of density functions of continuous random variables such as the Gamma distribution and Beta distribution.

Theorem. *For $p > 0$, the integral*

$$\int_0^\infty u^{p-1}\, e^{-u}\, du$$

is absolutely convergent.

Proof. Since we are considering the value of the improper integral

$$\int_0^\infty u^{p-1}\, e^{-u}\, du$$

for all $p > 0$, there is need to be careful at both endpoints 0 and ∞.

We begin with the easiest case. If $p = 1$, then

$$\int_0^\infty u^0 e^{-u}\, du = \int_0^\infty e^{-u}\, du = \lim_{N\to\infty} \int_0^N e^{-u}\, du = \lim_{N\to\infty} (1 - e^{-N}) = 1.$$

For the remaining cases $0 < p < 1$ and $p > 1$ we will consider the integral from 0 to 1 and the integral from 1 to ∞ separately.

If $0 < p < 1$, then the integral

$$\int_0^1 u^{p-1}\, e^{-u}\, du$$

is improper. Thus,

$$\int_0^1 u^{p-1}\, e^{-u}\, du = \lim_{a\to 0+} \int_a^1 u^{p-1}\, e^{-u}\, du \le \lim_{a\to 0+} \int_a^1 u^{p-1}\, du = \lim_{a\to 0+} \frac{1-a^p}{p} = \frac{1}{p}$$

since $e^{-u} \le 1$ for $0 \le u \le 1$.

Furthermore, if $0 < p < 1$, then $0 < u^{p-1} \le 1$ for $u \ge 1$ and so

$$\int_1^\infty u^{p-1}\, e^{-u}\, du = \lim_{N\to\infty} \int_1^N u^{p-1}\, e^{-u}\, du \le \lim_{N\to\infty} \int_1^N e^{-u}\, du = \lim_{N\to\infty} (1 - e^{-N}) = 1.$$

Thus, we can conclude that for $0 < p < 1$,

$$\int_0^\infty u^{p-1}\, e^{-u}\, du = \int_0^1 u^{p-1}\, e^{-u}\, du + \int_1^\infty u^{p-1}\, e^{-u}\, du \le \frac{1}{p} + 1 < \infty.$$

If $p > 1$, then $u^{p-1} \in [0,1]$ and $e^{-u} \leq 1$ for $0 \leq u \leq 1$. Thus,

$$\int_0^1 u^{p-1}\, e^{-u}\, du \leq \int_0^1 u^{p-1}\, du = \left.\frac{u^p}{p}\right|_0^1 = \frac{1}{p}.$$

On the other hand, if $p > 1$, let $\lfloor p \rfloor$ denote the smallest integer less than or equal to p so that $p - \lfloor p \rfloor \in [0,1)$. Thus, $0 < u^{p-\lfloor p \rfloor - 1} \leq 1$ for $u \geq 1$. We then have

$$\int_1^N u^{p-1}\, e^{-u}\, du = \int_1^N u^{p-\lfloor p \rfloor-1} u^{\lfloor p \rfloor}\, e^{-u}\, du \leq \int_1^N u^{\lfloor p \rfloor}\, e^{-u}\, du.$$

Integration by parts $\lfloor p \rfloor$ times (the so-called *reduction formula*) gives

$$\begin{aligned}
&\int_1^N u^{\lfloor p \rfloor}\, e^{-u}\, du \\
&\qquad = -e^{-u}\left(u^{\lfloor p \rfloor} + \lfloor p \rfloor u^{\lfloor p \rfloor - 1} + \lfloor p \rfloor \cdot (\lfloor p \rfloor - 1) u^{\lfloor p \rfloor - 2} + \cdots + \lfloor p \rfloor \cdot (\lfloor p \rfloor - 1) \cdots 2 \cdot u\right)\Big|_1^N \\
&\qquad\qquad + \lfloor p \rfloor \cdot (\lfloor p \rfloor - 1) \cdots 2 \cdot 1 \cdot \int_1^N e^{-u}\, du
\end{aligned}$$

and so

$$\lim_{N \to \infty} \int_1^N u^{\lfloor p \rfloor}\, e^{-u}\, du = \lfloor p \rfloor\,!.$$

Thus, we can conclude that for $p > 1$,

$$\int_0^\infty u^{p-1}\, e^{-u}\, du = \int_0^1 u^{p-1}\, e^{-u}\, du + \int_1^\infty u^{p-1}\, e^{-u}\, du \leq \frac{1}{p} + \lfloor p \rfloor\,! < \infty.$$

In every case we have $u^{p-1}\, e^{-u} \geq 0$ and so

$$\int_0^\infty \left|u^{p-1}\, e^{-u}\right|\, du = \int_0^\infty u^{p-1}\, e^{-u}\, du < \infty.$$

That is, this integral is absolutely convergent, and so $\Gamma(p)$ is well-defined for $p > 0$. $\square$