Math 302.102 Fall 2010
Solutions to Assignment #3

1. Let A be the event that a randomly selected person is a man so that A^c is the event that a randomly selected person is a woman. Let B be the event that a person is colour blind. We are told that $\mathbf{P}\{B \mid A\} = 0.05$ and $\mathbf{P}\{B \mid A^c\} = 0.0025$.

(a) If men and women each make up the same proportion of the population, then $\mathbf{P}\{A\} = 0.5$ and $\mathbf{P}\{A^c\} = 0.5$, so that Bayes' Rule tells us that

$$\mathbf{P}\{A \mid B\} = \frac{\mathbf{P}\{B \mid A\}\mathbf{P}\{A\}}{\mathbf{P}\{B\}} = \frac{\mathbf{P}\{B \mid A\}\mathbf{P}\{A\}}{\mathbf{P}\{B \mid A\}\mathbf{P}\{A\} + \mathbf{P}\{B \mid A^c\}\mathbf{P}\{A^c\}} = \frac{(0.05)(0.5)}{(0.05)(0.5) + (0.0025)(0.5)}$$
$$= \frac{20}{21} \doteq 0.952381.$$

(b) If there are twice as many women as men in this population, then $\mathbf{P}\{A\} = 1/3$, $\mathbf{P}\{A^c\} = 2/3$, and Bayes' Rule tells us that

$$\mathbf{P}\{A \mid B\} = \frac{\mathbf{P}\{B \mid A\}\mathbf{P}\{A\}}{\mathbf{P}\{B\}} = \frac{\mathbf{P}\{B \mid A\}\mathbf{P}\{A\}}{\mathbf{P}\{B \mid A\}\mathbf{P}\{A\} + \mathbf{P}\{B \mid A^c\}\mathbf{P}\{A^c\}} = \frac{(0.05)(1/3)}{(0.05)(1/3) + (0.0025)(2/3)}$$
$$= \frac{10}{11} \doteq 0.909091.$$

2. Let A be the event that a randomly selected chip is good so that A^c is the event that a randomly selected chip is bad. Let B be the event that a chip passes the cheap chip test. We are told that $\mathbf{P}\{A\} = 0.8$ and $\mathbf{P}\{A^c\} = 0.2$. Since all good chips pass the cheap chip test, $\mathbf{P}\{B \mid A\} = 1$, but since 10% of bad chips also pass the cheap chip test, we have $\mathbf{P}\{B \mid A^c\} = 0.1$.

(a) Using Bayes' Rule we find

$$\mathbf{P}\{A \mid B\} = \frac{\mathbf{P}\{B \mid A\}\mathbf{P}\{A\}}{\mathbf{P}\{B\}} = \frac{\mathbf{P}\{B \mid A\}\mathbf{P}\{A\}}{\mathbf{P}\{B \mid A\}\mathbf{P}\{A\} + \mathbf{P}\{B \mid A^c\}\mathbf{P}\{A^c\}} = \frac{(1)(0.8)}{(1)(0.8) + (0.1)(0.2)}$$
$$= \frac{40}{41} \doteq 0.975609.$$

(b) If the company sells all chips which pass the cheap chip test, then the percentage of chips sold that are bad is simply

$$\mathbf{P}\{A^c \mid B\} = 1 - \mathbf{P}\{A \mid B\} = 1 - \frac{40}{41} = \frac{1}{41}.$$

3. Let A be the event that a randomly selected person has the disease so that A^c is the event that a randomly selected person does not have the disease. Let B be the event that that the laboratory test on the blood sample returns a positive result. We are told that $\mathbf{P}\{A\} = 0.01$ so that $\mathbf{P}\{A^c\} = 0.99$. We are also told that $\mathbf{P}\{B \mid A\} = 0.95$ and $\mathbf{P}\{B \mid A^c\} = 0.02$. Using Bayes' Rule we find

$$\mathbf{P}\{A \mid B\} = \frac{\mathbf{P}\{B \mid A\}\mathbf{P}\{A\}}{\mathbf{P}\{B\}} = \frac{\mathbf{P}\{B \mid A\}\mathbf{P}\{A\}}{\mathbf{P}\{B \mid A\}\mathbf{P}\{A\} + \mathbf{P}\{B \mid A^c\}\mathbf{P}\{A^c\}} = \frac{(0.95)(0.01)}{(0.95)(0.01) + (0.02)(0.99)}$$
$$= \frac{95}{293} \doteq 0.324232.$$

4. The key to solving this problem is to realize that the event A from the previous problem needs to change. No longer is the patient who walks into the doctor's office "randomly selected from the population." Hence, we take A to be the event that the patient has the disease. The doctor's opinion is that $\mathbf{P}\{A\} = 0.3$. If the blood test result is positive, then Bayes' Rule implies

$$\mathbf{P}\{A \,|\, B\} = \frac{\mathbf{P}\{B \,|\, A\}\,\mathbf{P}\{A\}}{\mathbf{P}\{B\}} = \frac{\mathbf{P}\{B \,|\, A\}\,\mathbf{P}\{A\}}{\mathbf{P}\{B \,|\, A\}\,\mathbf{P}\{A\} + \mathbf{P}\{B \,|\, A^c\}\,\mathbf{P}\{A^c\}} = \frac{(0.95)(0.30)}{(0.95)(0.30) + (0.02)(0.70)}$$
$$= \frac{285}{299} \doteq 0.953177.$$

5. For $j = 1, 2, \ldots$, let A_j be the event that a 3 appears on roll j, let B_j be the event that a 1 or a 6 appears on roll j, let C_j be the event that a 2 appears on roll j, let D_j be the event that a 4 appears on roll j, and let E_j be the event that a 5 appears on roll j. If W denotes the event that Player A wins, then the Law of Total Probability implies

$$\mathbf{P}\{W\} = \mathbf{P}\{W \,|\, A_1\}\,\mathbf{P}\{A_1\} + \mathbf{P}\{W \,|\, B_1\}\mathbf{P}\{B_1\} + \mathbf{P}\{W \,|\, C_1\}\,\mathbf{P}\{C_1\}$$
$$+ \mathbf{P}\{W \,|\, D_1\}\,\mathbf{P}\{D_1\} + \mathbf{P}\{W \,|\, E_1\}\,\mathbf{P}\{E_1\}$$

Since Player A wins immediately if a 3 appears on the first roll, $\mathbf{P}\{W \,|\, A_1\} = 1$, and since Player A loses immediately if a 1 or a 6 appears on the first roll, $\mathbf{P}\{W \,|\, B_1\} = 0$. Furthermore, since the die is fair, $\mathbf{P}\{A_1\} = \mathbf{P}\{C_1\} = \mathbf{P}\{D_1\} = \mathbf{P}\{E_1\} = 1/6$. Therefore,

$$\mathbf{P}\{W\} = \frac{1}{6} + \frac{1}{6}\mathbf{P}\{W \,|\, C_1\} + \frac{1}{6}\mathbf{P}\{W \,|\, D_1\} + \frac{1}{6}\mathbf{P}\{W \,|\, E_1\}.$$

The next observation is that $\mathbf{P}\{W \,|\, C_1\} = \mathbf{P}\{W \,|\, D_1\} = \mathbf{P}\{W \,|\, E_1\}$. If Player A rolls a 2 initially, then the only way for Player A to win is if a 2 is rolled before a 3 on subsequent rolls. Similarly, if Player A rolls a 4 initially, then the only way for Player A to win is if a 4 is rolled before a 3 on subsequent rolls, and if Player A rolls a 5 initially, then the only way for Player A to win is if a 5 is rolled before a 3 on subsequent rolls. Since the die is fair, all of these events have the same probability. Thus,

$$\mathbf{P}\{W\} = \frac{1}{6} + \frac{1}{2}\mathbf{P}\{W \,|\, C_1\}.$$

The remaining step is to compute $\mathbf{P}\{W \,|\, C_1\}$. It is reasonable to guess that $\mathbf{P}\{W \,|\, C_1\} = 1/2$. This is because if a 2 is rolled initially, then the only way for Player A to win is if a 2 appears before a 3 on subsequent rolls. Since only a 2 or 3 matter (all other numbers cause a re-roll), and since 2 and 3 are both equally likely, we must have $\mathbf{P}\{W \,|\, C_1\} = 1/2$. Alternatively, we can sum up an infinite series as follows.

(continued)

Let $F_j = B_j \cap D_j \cap E_j$ be the event that 1, 4, 5, or 6 appears on roll j. Then, using the fact that the results of subsequent rolls are independent, we find

$$\begin{aligned}
\mathbf{P}\{W \mid C_1\} &= \mathbf{P}\{C_2\} + \mathbf{P}\{F_2\}\mathbf{P}\{C_3\} + \mathbf{P}\{F_2\}\mathbf{P}\{F_3\}\mathbf{P}\{C_4\} + \cdots \\
&= \frac{1}{6} + \frac{4}{6}\cdot\frac{1}{6} + \left(\frac{4}{6}\right)^2 \cdot \frac{1}{6} + \cdots \\
&= \frac{1}{6}\left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right] \\
&= \frac{1}{6}\cdot\frac{1}{1-2/3} \\
&= \frac{1}{2}.
\end{aligned}$$

This implies that

$$\mathbf{P}\{W\} = \frac{1}{6} + \frac{1}{2}\cdot\frac{1}{2} = \frac{5}{12},$$

and so the probability that Player A loses is

$$\mathbf{P}\{W^c\} = 1 - \frac{5}{12} = \frac{7}{12}.$$

6. **(a)** Since

$$\int_1^\infty x^{-2}\,\mathrm{d}x = -x^{-1}\Big|_1^\infty = 0 - (-1) = 1$$

we see that taking $c = 1$ makes f a legitimate probability density.

(b) Since

$$\int_1^\infty x^{-1}\,\mathrm{d}x = \ln(x)\Big|_1^\infty = \infty - 0 = \infty$$

we see that there is no such c that makes f a legitimate probability density.

(c) Using integration-by-parts twice (see Prerequisite Review Handout) gives

$$\int x^2 e^{-x}\,\mathrm{d}x = -x^2e^{-x} - 2xe^{-x} - 2e^{-x}.$$

Therefore, since

$$\int_{-1}^1 x^2e^{-x}\,\mathrm{d}x = \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]\Big|_{-1}^1 = [-e^{-1} - 2e^{-1} - 2e^{-1}] - [-e^1 + 2e^1 - 2e^1] = e - 5e^{-1},$$

we see that taking $c = (e - 5e^{-1})^{-1} = e/(e^2 - 5)$ makes f a legitimate probability density.

(d) As in **(c)**, using integration-by-parts twice gives

$$\int_0^\infty x^2e^{-x}\,\mathrm{d}x = \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]\Big|_0^\infty = 0 - (-2) = 2$$

and so we see that taking $c = 1/2$ makes f a legitimate probability density.

(e) Since

$$\int_{-\infty}^{0} xe^x \,\mathrm{d}x = [xe^x - e^x]\Big|_{-\infty}^{0} = (0-1)-(0-0) = -1$$

by see that taking $c = -1$ makes f a legitimate probability density. Note that $xe^x \le 0$ for $x \le 0$. This means that if we multiply xe^x by a negative number it will always be non-negative. Hence, $f(x) = -xe^x$ for $x \le 0$ is a non-negative function that integrates to 1.

(f) Note that the function xe^x assumes both positive and negative values when $-1 \le x \le 1$. This means that there is no single value of c that could be multiplied by xe^x to make it strictly non-negative. Hence, there is no possible value of c that makes f a legitimate probability density.