Math 302.102 Fall 2010
Assignment #6

This assignment is due at the beginning of class on Wednesday, November 10, 2010.

1. The exponential distribution has an important property that uniquely characterizes it among continuous distributions, the *lack of memory property*, also known as the *memoryless property*. Suppose that $X \sim \text{Exp}(\lambda)$ and let $s > 0$ and $t > 0$ be real numbers. Show, by a direct calculation, that

$$\mathbf{P}\{X > t + s \,|\, X > t\} = \mathbf{P}\{X > s\}.$$

In other words, start with the object on the left side of the equality, manipulate it using the definition of conditional probability, and arrive at the expression on the right side of the equality. What this identity says is that if the lifetime of a component follows an exponential distribution, then the probability that the component's lifetime is at least $s + t$ given that the component's lifetime is at least t is simply the probability that the component's lifetime is at least s. An alternative interpretation is the following. Suppose you are standing in line and the amount of time in minutes until you are served is exponentially distributed. If you have already stood in line for at least t minutes, then the amount of time remaining until you will actually be served is independent of the time already spent in line.

Note: It is a much harder problem (not Math 302) to show that if X is a continuous random variable such that $\mathbf{P}\{X > t + s \,|\, X > t\} = \mathbf{P}\{X > s\}$, then $X \sim \text{Exp}(\lambda)$ for any $\lambda > 0$. Essentially the argument boils down to showing that the only decreasing, continuous functions $h : [0, \infty) \to [0, \infty)$ with the property that $h(s+t) = h(s) + h(t)$ are of the form $h(t) = -\lambda t$ where $\lambda > 0$ is a constant. (This is known as Cauchy's functional equation; see Wikipedia for more information.)

2. Suppose that $X \sim \text{Unif}(0, 1)$.

(a) Show that $\mu = \mathbb{E}(X) = \dfrac{1}{2}$ and $\sigma^2 = \text{var}(X) = \dfrac{1}{12}$.

(b) Compute $\mathbf{P}\{\mu - 2\sigma < X < \mu + 2\sigma\}$. How does this compare to the estimate promised by Chebychev's inequality?

3. Suppose that X is continuous random variable with density function

$$f_X(x) = \frac{\Gamma(\frac{m+n}{2})\left(\frac{m}{n}\right)^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \frac{x^{m/2-1}}{(1 + \frac{mx}{n})^{(m+n)/2}}$$

for $0 < x < \infty$ where m and n are positive integers. Compute the density function of

$$Y = \frac{1}{1 + \frac{m}{n}X}$$

and verify that $Y \sim \text{Beta}(n/2, m/2)$.

4. Suppose that X is a continuous random variable with density function

$$f_X(x) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\,\Gamma(\frac{n}{2})} \cdot \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2}$$

for $-\infty < x < \infty$. Determine the density function of $Y = X^2$.

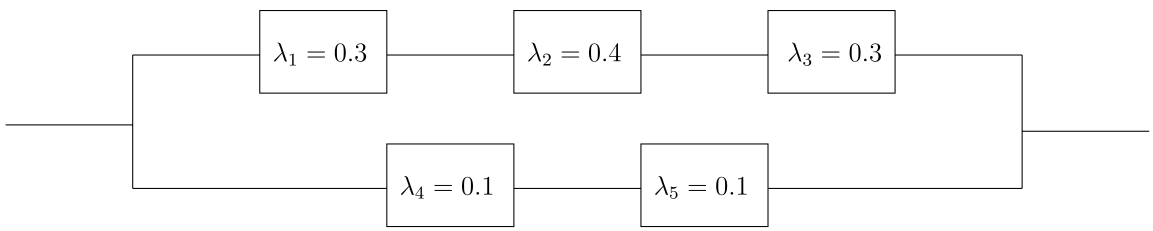
5. An earthquake of magnitude M releases energy X such that $M = \log X$. For earthquakes of magnitude greater than 3, suppose that $M - 3$ has an exponential distribution with mean 2.

(a) Determine $\mathbb{E}(M)$ and $\mathrm{Var}(M)$ for an earthquake of magnitude greater than 3.

(b) For an earthquake as in **(a)**, find the density of X.

(c) Consider two earthquakes, both of magnitude greater than 3. What is the probability that the magnitude of the smaller earthquake is greater than 4? Assume that the magnitudes of the two earthquakes are independent of each other.

6. An electrical circuit consists of five components connected as shown in the following diagram. The lifetimes of the components, measured in days, have independent exponential distributions with parameters λ_i, $i = 1, 2, 3, 4, 5$, as indicated. The circuit will continue to operate as long as there is at least one path from left-to-right consisting of functioning components. Determine the expected lifetime of the circuit.



7. If $X \sim \mathrm{Gamma}(\alpha, \lambda)$, compute the moment generating function $m(t) = \mathbb{E}(e^{tX})$ of X.