Stat 862 (Winter 2007)
Method of Relaxations

We now explain a simple numerical algorithm for constructing solutions to the discrete Dirichlet problem. The **method of relaxations** was originally developed as a means of approximating solutions to the original (continuous) Dirichlet problem. However, this method is extremely well-suited for the discrete problem. See [1] and the references therein for more information.

1. Consider a discrete set A with boundary ∂A.

2. Assign boundary values $f(x)$ for $x \in \partial A$.

3. Begin by assigning the value 0 to all $x \notin \partial A$.

4. Systematically proceed through all non-boundary points by starting at one interior point and adjusting the value there to be equal to the average value of its neighbours.

5. Continue through all the interior points updating each point by the average of its neighbours.

6. After one iteration of this process, the result will not be harmonic, although it will be more harmonic than the initial values.

7. Iterate this process and each time the resulting functions will better approximate the solution to the Dirichlet problem.

Example. *Consider the set of points in $\mathbb{Z}^2$ in the 5×5 grid about the origin. Suppose that we assign value 1 to all 16 exterior boundary points and boundary value 0 to the origin. Then the Dirichlet problem requires us to find the harmonic function on the interior and agreeing with the boundary values.*

1	1	1	1	1
1	*	*	*	1
1	*	0	*	1
1	*	*	*	1
1	1	1	1	1

After the first iteration of the method of relaxations, we have

1	1	1	1	1
1	.500	.375	.59375	1
1	.375	0	.3984375	1
1	.59375	.3984375	.69921875	1
1	1	1	1	1

where

$$\begin{aligned}
.500 &= \tfrac{1}{4}(1+1+0+0)\\
.375 &= \tfrac{1}{4}(1+.500+0+0)\\
.59375 &= \tfrac{1}{4}(1+.375+1+0)\\
.375 &= \tfrac{1}{4}(1+.500+0+0)\\
.3984375 &= \tfrac{1}{4}(.593751+1+0+0)\\
.59375 &= \tfrac{1}{4}(.375+1+1+0)\\
.3984375 &= \tfrac{1}{4}(.593751+1+0+0)\\
.69921875 &= \tfrac{1}{4}(.3984375+.3984375+1+1).
\end{aligned}$$

After several iterations it is clear that our values for the interior points have converged to the required harmonic function.

1	1	1	1	1
1	.833333	.666667	.833333	1
1	.666667	0	.666667	1
1	.833333	.666667	.833333	1
1	1	1	1	1

Thus, the solution to this Dirichlet problem is precisely

$$\begin{array}{ccccc}
1 & 1 & 1 & 1 & 1\\
1 & \frac{5}{6} & \frac{2}{3} & \frac{5}{6} & 1\\
1 & \frac{2}{3} & 0 & \frac{2}{3} & 1\\
1 & \frac{5}{6} & \frac{2}{3} & \frac{5}{6} & 1\\
1 & 1 & 1 & 1 & 1
\end{array}$$

We finally end this section with a statement that the method of relaxations algorithm converges to the harmonic function $u(x)$ with boundary values $f(x)$.

Theorem. *Suppose that the method of relaxations is started with an initial guess which has the property that the value at each point is* $\leq$ *the average of the values of the neighbours of this point. Then the successive values at a point* x *are monotonically increasing with a limit* $u(x)$ *and these limits provide a solution to the Dirichlet problem.*

References

[1] P. G. Doyle and J. L. Snell. *Random Walks and Electric Networks.* Mathematical Association of America, Washington, DC, 1984.