Stat 862 (Winter 2007)
Conditional Expectation

Claim. If A is an event of the form $A = \{a \le X \le b\}$, then $\mathbb{E}(\mathbb{E}(Y|X)1_A) = \mathbb{E}(Y1_A)$.

Proof. Since $\mathbb{E}(\mathbb{E}(Y|X))$ is X-measurable, we can write $\mathbb{E}(\mathbb{E}(Y|X)) = \varphi(X)$ for some function φ. Thus, by definition of expectation (in the continuous case)

$$\mathbb{E}(\mathbb{E}(Y|X)1_A) = \mathbb{E}(\varphi(X)1_A) = \int_{-\infty}^{\infty} \varphi(x)1_A(x)\, f_X(x)\, dx = \int_a^b \varphi(x)\, f_X(x)\, dx.$$

However, by definition of marginal density

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

and by definition of conditional expectation

$$\varphi(x) = \mathbb{E}(Y|X=x) = \frac{\int_{-\infty}^{\infty} y f(x,y)\, dy}{\int_{-\infty}^{\infty} f(x,y)\, dy} = \frac{\int_{-\infty}^{\infty} z f(x,z)\, dz}{\int_{-\infty}^{\infty} f(x,y)\, dy}$$

where we have used the dummy variable z instead of y in the last line in anticipation of the next step. Substituting in gives

$$\begin{aligned}\mathbb{E}(\mathbb{E}(Y|X)1_A) = \int_a^b \varphi(x)\, f_X(x)\, dx &= \int_a^b \left(\frac{\int_{-\infty}^{\infty} z f(x,z)\, dz}{\int_{-\infty}^{\infty} f(x,y)\, dy}\right) \int_{-\infty}^{\infty} f(x,y)\, dy\, dx \\ &= \int_a^b \int_{-\infty}^{\infty} z f(x,z)\, dz\, dx \\ &= \int_a^b \int_{-\infty}^{\infty} y f(x,y)\, dy\, dx \qquad (*)\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathbb{E}(Y1_A) = \int_{-\infty}^{\infty} y1_A(x) f_Y(y)\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y1_A(x) f(x,y)\, dx\, dy &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y1_A(x) f(x,y)\, dy\, dx \\ &= \int_{-\infty}^{\infty} 1_A(x) \int_{-\infty}^{\infty} y f(x,y)\, dy\, dx \\ &= \int_a^b \int_{-\infty}^{\infty} y f(x,y)\, dy\, dx. \quad (**)\end{aligned}$$

Thus, comparing $(*)$ and $(**)$ we conclude that

$$\mathbb{E}(\mathbb{E}(Y|X)1_A) = \mathbb{E}(Y1_A)$$

as required. □