Stat 862 (Winter 2007)
Markov Chain Monte Carlo Methods

Suppose that $T$ is the set of $N \times N$ matrices whose entries are either 0 or 1 and such that no two 1s are adjacent. If $N = 2$, then $T$ consists of the following 7 matrices:

$$T = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}.$$

If we label the seven elements of $T$ by $\{1, 2, 3, 4, 5, 6, 7\}$, then we can define a symmetric, irreducible, aperiodic Markov chain on $T$ via the following transition matrix:

$$\mathbf{P} = \begin{bmatrix} 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 \\ 1/4 & 1/2 & 0 & 0 & 0 & 1/4 & 0 \\ 1/4 & 0 & 1/2 & 0 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 1/2 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 0 & 1/2 & 1/4 & 0 \\ 0 & 1/4 & 0 & 0 & 1/4 & 1/2 & 0 \\ 0 & 0 & 1/4 & 1/4 & 0 & 0 & 1/2 \end{bmatrix}.$$

Note that $\mathbf{P}$ does, in fact, define a symmetric, irreducible, aperiodic Markov chain on $T$.

We can now use the methods of Chapter 1 to analyze the long-run behaviour of this Markov chain. Since the Markov chain is irreducible and aperiodic, there exists a unique invariant probability vector $\bar{\pi}$ satisfying $\bar{\pi}\mathbf{P} = \bar{\pi}$ and

$$\lim_{n \to \infty} \bar{\phi}_0 \mathbf{P}^n = \bar{\pi}$$

for any initial probability vector $\bar{\phi}_0$.

Solving the $7 \times 7$ system by hand is possible, though tedious, and gives

$$\bar{\pi} = \left( \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7} \right)$$

which is to be expected.

Using Maple to compute high powers of $\mathbf{P}$ we find that

- $\mathbf{P}^{30}$ gives $\bar{\pi} \approx (0.14, 0.14, 0.14, 0.14, 0.14, 0.14, 0.14)$ which is accurate to 2 decimal places,
- $\mathbf{P}^{40}$ gives $\bar{\pi} \approx (0.143, 0.143, 0.143, 0.143, 0.143, 0.143, 0.143)$ which is accurate to 3 decimal places,
- $\mathbf{P}^{100}$ gives $\bar{\pi} \approx (0.142857, 0.142857, 0.142857, 0.142857, 0.142857, 0.142857, 0.142857)$ which is accurate to 6 decimal places.