Stat 862: Stochastic Process
Winter 2006

Exercise: Suppose that S and T are stopping times. Show that $\mathcal{F}_S \cap \mathcal{F}_T = \mathcal{F}_{S \wedge T}$.

Solution: Suppose that $A \in \mathcal{F}_S \cap \mathcal{F}_T$ so that $A \cap \{S \leq n\} \in \mathcal{F}_n$ and $A \cap \{T \leq n\} \in \mathcal{F}_n$ for each n. Since $\{S \wedge T \leq n\} = \{S \leq n\} \cup \{T \leq n\}$, we find

$$A \cap \{S \wedge T \leq n\} = A \cap [\{S \leq n\} \cup \{T \leq n\}] = [A \cap \{S \leq n\}] \cup [A \cap \{T \leq n\}] \in \mathcal{F}_n$$

so that $A \in \mathcal{F}_{S \wedge T}$. Thus, $\mathcal{F}_S \cap \mathcal{F}_T \subseteq \mathcal{F}_{S \wedge T}$. Conversely, suppose that $A \in \mathcal{F}_{S \wedge T}$. This implies that $A \cap \{S \wedge T \leq n\} \in \mathcal{F}_n$ for each n so that

$$A \cap \{S \wedge T \leq n\} = A \cap [\{S \leq n\} \cup \{T \leq n\}] = [A \cap \{S \leq n\}] \cup [A \cap \{T \leq n\}] \in \mathcal{F}_n.$$

However,

$$[A \cap \{S \leq n\}] \subseteq [A \cap \{S \leq n\}] \cup [A \cap \{T \leq n\}] \in \mathcal{F}_n$$

so that $A \in \mathcal{F}_S$, and similarly

$$[A \cap \{T \leq n\}] \subseteq [A \cap \{S \leq n\}] \cup [A \cap \{T \leq n\}] \in \mathcal{F}_n$$

so that $A \in \mathcal{F}_T$. Therefore, $A \in \mathcal{F}_S \cap \mathcal{F}_T$, and we conclude $\mathcal{F}_{S \wedge T} \subseteq \mathcal{F}_S \cap \mathcal{F}_T$. Taken together, we have $\mathcal{F}_{S \wedge T} = \mathcal{F}_S \cap \mathcal{F}_T$ as required.