University of Regina
Department of Mathematics & Statistics
Final Examination (PART I)
200610

Statistics 862
Stochastic Processes

Instructor: Michael Kozdron

This final exam consists of two parts.

Part I of this exam is due at 1:00 pm on Friday, April 21, 2006, in my office (College West 307.31).

Part II of this exam will begin at 2:00 pm on Friday, April 21, 2006, in Classroom Building 251 (CL 251).

Read all of the following information before starting Part I of the exam.

Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit.

You may use standard notation; however, any new notations or abbreviations that you introduce must be clearly defined. You may cite results from our textbook Probability Essentials, *by P. Protter and J. Jacod, without recopying them.*

You may consult any notes or books or websites that you wish, provided that proper acknowledgements are included. However, you may not discuss this exam with anyone before 1:00 pm on April 21, 2006. This includes students, professors, and colleagues.

*Part I of this exam has **9** problems, and it is worth a total of **90** points.*

*Part II of this exam has **4** problems, and it is worth a total of **60** points.*

1. (*10 points*) Let $B = \{B_t,\ t \ge 0\}$ be a standard Brownian motion with $B_0 = 0$.

(a) Compute $P\left(\max_{1\le t\le 2} B_t \ge 0 \,\middle|\, B_1 < 0\right)$.

(b) Compute $P\left(\max_{1\le t\le 2} B_t \ge 0\right)$.

2. (*10 points*) Suppose that $g : [0,1] \to \mathbb{R}$ is a step function in $L^2[0,1]$, and define the process $I = \{I_t,\ 0 \le t \le 1\}$ by setting

$$I_t = \int_0^t g(s)\, dB_s.$$

Show directly that I is a martingale (with respect to the Brownian filtration).

Hint: For $0 \le s \le t \le 1$, write

$$\mathbb{E}(I_t|\mathcal{F}_s) = \mathbb{E}(I_s|\mathcal{F}_s) + \mathbb{E}\left(\int_s^t g(u)\, dB_u \middle| \mathcal{F}_s\right),$$

and show that

$$\mathbb{E}\left(\int_s^t g(u)\, dB_u \middle| \mathcal{F}_s\right) = 0$$

by considering the function g restricted to the interval $[s,t]$.

3. (*10 points*) Suppose that $B = \{B_t,\ 0 \le t \le 1\}$ is a standard Brownian motion with $B_0 = 0$, and let

$$g(t) = B_{\frac{1}{2}} \cdot \mathbb{1}_{[\frac{1}{2},1]}(t).$$

Show that

$$\int_0^1 g(t)\, dB_s$$

is NOT a normal random variable.

Hint: Explain why the product of normal random variables need not be normal. You might want to consider taking logs of the product and using moment generating functions. (See also Section 4.3 in Casella and Berger).

4. (*10 points*) Suppose that $B = \{B_t,\ t \ge 0\}$ is a standard Brownian motion with $B_0 = 0$, and that the process $X = \{X_t,\ t \ge 0\}$ is defined by

$$X_t = B_t - \frac{3}{t}\int_0^t B_s\, ds.$$

Show that X is a standard Brownian motion.

Hint: Write $B_t = \displaystyle\int_0^t dB_s$.

5. (*10 points*) Suppose that $B = \{B_t,\ t \ge 0\}$ is a standard Brownian motion with $B_0 = 0$, and that the process $X = \{X_t,\ t \ge 0\}$ is defined by

$$X_t = \frac{1}{3}B_t^3 - \int_0^t B_s\, ds.$$

Show that X is a martingale (with respect to the Brownian filtration).

Problems 6–9 discuss the discrete versions of the classic partial differential equations that we studied in class, and use the following notation.

Let $\mathbb{Z}^2 = \{(x, y) : x, y \text{ are integers}\}$, and let $e_1 = (0, 1)$, $e_2 = (1, 0)$ be the unit vectors. If $f : \mathbb{Z}^2 \to \mathbb{R}$ is a function and $x \in \mathbb{R}$, we define the *discrete Laplacian of f at x* by

$$\Delta f(x) = \frac{1}{4}\left[f(x + e_1) + f(x - e_1) + f(x + e_2) + f(x - e_2)\right] - f(x).$$

We say that f *is harmonic at x* if $\Delta f(x) = 0$. We write $S = \{S_n,\ n = 0, 1, \dots\}$ to denote simple random walk on $\mathbb{Z}^2$; that is,

$$S_n = S_0 + X_1 + X_2 + \cdots + X_n$$

where $X_1, \dots, X_n$ are independent and identically distributed random variables with

$$P(X_j = e_1) = P(X_j = -e_1) = P(X_j = e_2) = P(X_j = -e_2) = \frac{1}{4}.$$

We use P^x and $\mathbb{E}^x$ to denote probabilities and expectations assuming $S_0 = x$. We let $\mathcal{F}_n$ be the σ-algebra generated by $X_1, X_2, \dots, X_n$. Let A be a finite connected subset of $\mathbb{Z}^2$. (By *connected*, we mean that there is a path from any two points in A staying in A.) Let

$$\partial A = \{x \in \mathbb{Z}^2 \setminus A : |x - y| = 1 \text{ for some } y \in A,$$

and write $\overline{A} = A \cup \partial A$. We say that f *is harmonic on A* if $\Delta f(x) = 0$ for every $x \in A$. Finally, let

$$\tau = \tau_A = \min\{n \geq 0 : S_n \notin A\}.$$

6. (*10 points*) Let $M_n = f(S_{n \wedge \tau})$. Show that M_n is a martingale (wrt the filtration $\{\mathcal{F}_n\}$) if and only if f is harmonic on A.

7. (*10 points*) Let $F : \partial A \to \mathbb{R}$ be given. Show that there is a unique function $f : \overline{A} \to \mathbb{R}$ such that f is harmonic on A and $f = F$ on ∂A. Give a probabilistic expression for $f(x)$.

8. (*10 points*) Let $g : A \to \mathbb{R}$ be given. Show that there is a unique function $f : \overline{A} \to \mathbb{R}$ such that $\Delta f(x) = -g(x)$ for $x \in A$ and $f(x) = 0$ for $x \in \partial A$. Give a probabilistic form of the solution.

Hint: Suppose that f is a solution. Show that

$$M_n = \sum_{j=0}^{(\tau \wedge n)-1} g(S_j)$$

is a martingale (wrt the filtration $\{\mathcal{F}_n\}$).

9. (*10 points*) We say that the function

$$u : \{0, 1, \dots\} \times \overline{A} \to \mathbb{R}$$

satisfies the heat equation with boundary condition F and initial condition f if

$$u(n + 1, x) - u(n, x) = \Delta u(n, x), \quad n \geq 0, \quad x \in A,$$

$$u(n, x) = F(x), \quad n \geq 0, \quad x \in \partial A,$$

$$u(0, x) = f(x), \quad x \in A.$$

Show that for any $F : \partial A \to \mathbb{R}$ and $f : A \to \mathbb{R}$ there is a unique solution to the heat equation. Give a probabilistic form of the solution. (Here Δ acts only on the x-variable.)

University of Regina
Department of Mathematics & Statistics
Final Examination (PART II)
200610
(April 21, 2005)

Statistics 862
Stochastic Processes

Name: ______________________ **Student Number:** ______________

Instructor: Michael Kozdron **Time:** 3 hours

This final exam consists of two parts.

Part I of this exam is due at 1:00 pm on Friday, April 21, 2006, in my office (College West 307.31).

Part II of this exam will begin at 2:00 pm on Friday, April 21, 2006, in Classroom Building 251 (CL 251).

Read all of the following information before starting Part II of the exam.

*You have **3** hours to complete Part II of this exam. Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit.* ***Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.*** *Points will be deducted for incoherent, incorrect, and/or irrelevant statements.*

You may use standard notation; however, any new notations or abbreviations that you introduce must be clearly defined.

This is a closed-book exam; no aids are permitted.

Note that blank space is not an indication of a question's difficulty. The order of the test questions is essentially random; they are not intentionally written easiest-to-hardest.

*Part I of this exam has **9** problems. Each problem is worth **10** points for a total of **90** points.*

*Part II of this exam has **4** problems, and it was worth a total of **60** points.*

DO NOT WRITE BELOW THIS LINE

Problem 10	______	Problem 11	______	Problem 12	______
Problem 13	______				
		Part I Total	______	Part II Total	______
				TOTAL	______

10. (*16 points*) Let $g : [0,1] \to \mathbb{R}$ be the step function

$$g(t) = 2 \cdot \mathbb{1}_{[0,\frac{1}{4})}(t) + 3 \cdot \mathbb{1}_{[\frac{1}{4},\frac{5}{8})}(t) + 7 \cdot \mathbb{1}_{[\frac{5}{8},\frac{3}{4})}(t) + 6 \cdot \mathbb{1}_{[\frac{3}{4},1]}(t).$$

(a) Sketch the graph of t vs. $g(t)$.

(b) Compute the Riemann-Stieltjes integral $\displaystyle\int_0^1 t^2 \, dg(t)$.

(c) Determine the distribution of the Wiener integral $\displaystyle\int_0^1 g(t) \, dB_t$.

11. (*8 points*) Suppose that $B = \{B_t,\ t \geq 0\}$ is a standard Brownian motion with $B_0 = 0$, and that the process $X = \{X_t,\ t \geq 0\}$ is defined by

$$X_t = \int_0^t e^{t-u}\, dB_u.$$

(a) For fixed $t > 0$, determine the distribution of the random variable X_t.

(b) For $s \leq t$, compute the covariance $\mathbb{E}(X_s X_t)$.

12. (*20 points*) This problem considers a version of the gambler's ruin problem for unfair games. Suppose that $0 < p < 1$ with $p \neq \frac{1}{2}$, and let $X_1, X_2, \ldots$ be independent and identically distributed random variables with

$$P(X_1 = 1) = p \quad \text{and} \quad P(X_1 = -1) = 1 - p.$$

Let $S = \{S_n,\ n = 0, 1, \ldots\}$ denote the random walk

$$S_n = S_0 + X_1 + \cdots + X_n.$$

Let $N \geq 1$ and x be integers with $0 \leq x \leq N$ and suppose that $S_0 = x$. Define the stopping time T by

$$T := \min\{j \geq 0 : S_j = 0 \quad \text{or} \quad S_j = N\}.$$

Compute

$$P^x\{S_T = N\} \quad \text{and} \quad \mathbb{E}^x(T).$$

Hint: Let

$$Y_n = \left(\frac{1-p}{p}\right)^{S_n} \quad \text{and} \quad Z_n = S_n - n(2p - 1).$$

Show that Y and Z are both martingales and use the optional sampling theorem. (You may assume that it is valid to apply the optional sampling theorem to Y and Z.)

13. (*16 points*) Suppose that B is a standard Brownian motion with $B_0 = 0$, and consider the sequence of stopping times $T_0, T_1, T_2, \ldots$ defined by setting $T_0 = 0$ and

$$T_{n+1} = \inf\{t > T_n : |B_t - B_{T_n}| = 1\}.$$

Define a sequence of random variables $S_0, S_1, S_2, \ldots$ by setting $S_n = B_{T_n}$. Verify that the process $S = \{S_n,\ n = 0, 1, 2, \ldots\}$ is a simple random walk starting at 0.

Hint: Let $X_0 = 0$, and for $n = 1, 2, \ldots$, let $X_n = S_n - S_{n-1} = B_{T_n} - B_{T_{n-1}}$ so that

$$S_n = \sum_{i=0}^{n} X_i.$$

Show that $X_1, X_2, \ldots$ are independent and identically distributed random variables with

$$P(X_1 = 1) = P(X_1 = -1) = \frac{1}{2}.$$