

Statistics 851 (Fall 2013)
Some More Basic Set Theory

Suppose that $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a function. The range of f is

$$\{y \in \mathbb{R}^m : f(x) = y \text{ for some } x \in \mathbb{R}^n\}.$$

Note that the range of f is a subset of \mathbb{R}^m . If $A \subseteq \mathbb{R}^n$, then the image of A under f is the subset of the range (and therefore a subset of \mathbb{R}^m) given by

$$f(A) = \{f(x) : x \in A\}.$$

If $B \subseteq \mathbb{R}^m$, then the preimage of B under f is the subset of the domain of f (and therefore a subset of \mathbb{R}^n) given by

$$f^{-1}(B) = \{x \in \mathbb{R}^n : f(x) \in B\}.$$

Example. Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is given by $f(x) = x^2$. Determine

- (i) the range of f ,
- (ii) the image of $A = [-2, 2]$ under f ,
- (iii) the image of $A = (-5, -3) \cup [1, 4]$ under f , and
- (iv) the preimage of $B = (9, 25)$ under f .

Solution.

- (i) The range of f is $[0, \infty)$.
- (ii) The image of $[-2, 2]$ is $f([-2, 2]) = [0, 4]$.
- (iii) The image of $(-5, -3) \cup [1, 4]$ is $f(((-5, -3) \cup [1, 4])) = (9, 25) \cup [1, 16] = [1, 25)$.
- (iv) The preimage of $(9, 25)$ is $f^{-1}((9, 25)) = (-5, -3) \cup (3, 5)$.

Exercise. Suppose that $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is *any* function. Show that

- (i) if $B \subseteq \mathbb{R}^m$, then $[f^{-1}(B)]^c = f^{-1}(B^c)$,
- (ii) if $B_1, B_2, \dots \subseteq \mathbb{R}^m$, then

$$f^{-1}\left(\bigcup_{j=1}^{\infty} B_j\right) = \bigcup_{j=1}^{\infty} f^{-1}(B_j),$$

and

- (iii) if $B_1, B_2, \dots \subseteq \mathbb{R}^m$, then

$$f^{-1}\left(\bigcap_{j=1}^{\infty} B_j\right) = \bigcap_{j=1}^{\infty} f^{-1}(B_j).$$

In particular, it follows from (iii) that if $B_1, B_2, \dots \subseteq \mathbb{R}^m$ are disjoint, then $f^{-1}(B_1), f^{-1}(B_2), \dots$ are disjoint.