Lecture #6: Construction of a Probability (Part I)

As we showed in Lecture #4, when the sample space is $\Omega = [0,1]$, it is not possible to construct a probability $\mathbf{P} : 2^\Omega \to [0,1]$ satisfying both $\mathbf{P}\{[a,b]\} = b-a$ for all $0 \le a \le b \le 1$, and $\mathbf{P}\{A \oplus r\} = \mathbf{P}\{A\}$ for all $A \subseteq [0,1]$ and $0 < r < 1$. In other words, it is not possible to define a uniform probability for *all* subsets of $[0,1]$. We will now begin the process of showing that it *is* possible to construct a uniform probability on $[0,1]$ provided that the σ-algebra is strictly smaller than 2^Ω.

There are a number of different approaches that one can take to constructing the uniform probability on $[0,1]$ equipped with the Borel σ-algebra $\mathcal{B}_1$. Our strategy will be to first prove the monotone class theorem and then show that as a corollary one can define a uniform probability on intervals by $\mathbf{P}\{[a,b]\} = b-a$ for all $0 \le a \le b \le 1$ and uniquely extend it to the Borel sets of $[0,1]$. This will take several lectures.

Suppose that Ω is a sample space. Recall from Lecture #1 that a σ-algebra is closed under both complements and countable unions. If we weaken the condition that it be closed under countable unions to just finite unions, then we have what is called an *algebra*. Note that you showed on Assignment #1 that every σ-algebra is necessarily an algebra, but that the converse need not be true.

Definition. Suppose that $\mathcal{F}_0$ is a collection of subsets of the sample space Ω. We say that $\mathcal{F}_0$ is an *algebra (of subsets of Ω)* if

(i) $\Omega \in \mathcal{F}_0$,

(ii) $A \in \mathcal{F}_0$ implies $A^c \in \mathcal{F}_0$, and

(iii) $A_1, A_2, \ldots, A_n \in \mathcal{F}_0$ implies $\bigcup_{j=1}^n A_j \in \mathcal{F}_0$ for every positive integer n.

As we remarked with the definition of σ-algebra, we will often say "let $\mathcal{F}_0$ be an algebra" understanding the "of subsets of Ω" part as implicit. We will now define what it means for a set function $\mathbf{P} : \mathcal{F}_0 \to [0,1]$ to be a probability when $\mathcal{F}_0$ is an algebra.

Definition. Suppose that Ω is the sample space, and let $\mathcal{F}_0$ be an algebra of subsets of Ω. A function $\mathbf{P} : \mathcal{F}_0 \to [0,1]$ is called a *probability* if

(i) $\mathbf{P}\{\Omega\} = 1$, and

(ii) if $A_1, A_2, \ldots \in \mathcal{F}_0$ are disjoint with $\bigcup_{j=1}^\infty A_j \in \mathcal{F}_0$, then

$$\mathbf{P}\left\{\bigcup_{j=1}^\infty A_j\right\} = \sum_{j=1}^\infty \mathbf{P}\{A_j\}.$$

Since $\mathcal{F}_0$ is only an algebra, it is not necessarily the case that $\bigcup_{j=1}^{\infty} A_j \in \mathcal{F}_0$. However, the requirement for $\mathbf{P} : \mathcal{F}_0 \to [0,1]$ to be a probability is that it is countably additive whenever it just so happens that $\bigcup_{j=1}^{\infty} A_j$ *is* in $\mathcal{F}_0$.

We will now outline the basic strategy for the general construction of a probability. We will begin with a probability $\mathbf{P} : \mathcal{F}_0 \to [0,1]$ defined on an algebra $\mathcal{F}_0$. We will then show that it is possible to define a probability $\mathbf{Q} : \mathcal{F} \to [0,1]$ on the generated σ-algebra $\mathcal{F} = \sigma(\mathcal{F}_0)$ whose restriction to $\mathcal{F}_0$ is $\mathbf{P}$; that is to say, we will construct a probability $\mathbf{Q} : \sigma(\mathcal{F}_0) \to [0,1]$ with $\mathbf{Q}\{A\} = \mathbf{P}\{A\}$ for all $A \in \mathcal{F}_0$. We will then show that $\mathbf{Q}$ is, in fact, unique.

Definition. A class $\mathcal{C}$ of subsets of Ω is *closed under finite intersections* if

$$\bigcap_{i=1}^{n} A_i \in \mathcal{C}$$

for every $n < \infty$ and for every $A_1, \ldots, A_n \in \mathcal{C}$.

Example 6.1. If $\Omega = \{1,2,3,4\}$ and

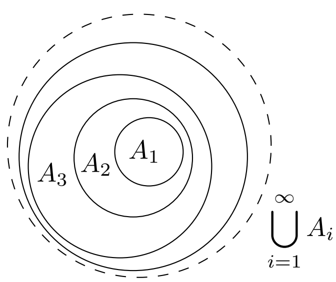
$$\mathcal{C} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\},$$

then $\mathcal{C}$ is not a σ-algebra (or even an algebra), but it is closed under finite intersections.

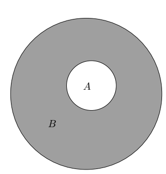
Definition. A class $\mathcal{C}$ of subsets of Ω is *closed under increasing limits* if

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{C}$$

for every collection $A_1, A_2, \ldots \in \mathcal{C}$ with $A_1 \subseteq A_2 \subseteq \cdots$.



Definition. A class $\mathcal{C}$ of subsets of Ω is *closed under finite differences* if for every $A, B \in \mathcal{C}$ with $A \subseteq B$, then $B \setminus A \in \mathcal{C}$.



We are now in a position to state the monotone class theorem.

Theorem 6.2 (Monotone Class Theorem)**.** *Let Ω be a sample space, and let $\mathcal{C}$ be a class of subsets of Ω. Suppose that $\mathcal{C}$ is closed under finite intersections and that $\mathcal{C}$ contains Ω (that is, $\Omega \in \mathcal{C}$). If $\mathcal{D}$ is the smallest class containing $\mathcal{C}$ which is closed under increasing limits and finite differences, then*

$$\mathcal{D} = \sigma(\mathcal{C}).$$

We will prove this theorem next lecture. We end with an example to show that the hypothesis $\Omega \in \mathcal{C}$ is vital.

Example 6.3. As in the previous example, suppose that $\Omega = \{1, 2, 3, 4\}$ and let

$$\mathcal{C} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

so that $\mathcal{C}$ is closed under finite intersections. Furthermore, $\mathcal{C}$ is also closed under increasing limits and finite differences as a simple calculation shows. Therefore, if $\mathcal{D}$ denotes the smallest class containing $\mathcal{C}$ which is closed under increasing limits and finite differences, then clearly $\mathcal{D} = \mathcal{C}$ itself. However, $\mathcal{C}$ is not a σ-algebra; the conclusion of the monotone class theorem suggests that $\mathcal{D} = \sigma(\mathcal{C})$ which would force $\mathcal{C}$ to be a σ-algebra. This is not a contradiction since the hypothesis that $\Omega \in \mathcal{C}$ is not met. Suppose that

$$\mathcal{C}' = \mathcal{C} \cup \Omega.$$

Then $\mathcal{C}'$ is still closed under finite intersections. However, it is no longer closed under finite differences. As a calculation shows, the smallest σ-algebra containing $\mathcal{C}'$ is now $\sigma(\mathcal{C}') = 2^{\Omega}$.