Statistics 452 Final Exam – December 9, 2011

This exam has 110 possible points but will be scored out of 100 points.

This exam has 6 problems and 3 numbered page.

You have 3 hours to complete this exam. Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit. ***Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.*** *Points will be deducted for incoherent, incorrect, and/or irrelevant statements.*

You are permitted to consult your class notes and the textbook ***Statistical Inference*** *by Casella and Berger.*

Instructor: Michael Kozdron

1. (*16 points*) Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed Bernoulli(θ) random variables where $0 \le \theta \le 1$ is a parameter and $n \ge 2$. Let

$$T = \sum_{i=1}^{n} X_i$$

so that T has a Binomial(n, θ) distribution. Show that

$$W(X_1, \ldots, X_n) = \frac{T(T-1)}{n(n-1)}$$

is the minimum variance unbiased estimator of θ^2.

2. (*24 points*) Suppose that $X_1, X_2, \ldots, X_n$ are independent random variables with

$$X_i \sim \mathcal{N}(i\theta, 1)$$

for $i = 1, \ldots, n$.

(a) Show that

$$\hat{\theta}(X_1, \ldots, X_n) = \frac{\sum_{i=1}^{n} iX_i}{\sum_{i=1}^{n} i^2}$$

is the maximum likelihood estimator of θ.

(b) Find the variance of $\hat{\theta}(X_1, \ldots, X_n)$.

(c) Compare this variance with the Cramér-Rao lower bound for unbiased estimation of θ.

3. (*24 points*) Let X_1 and X_2 be independent and identically distributed random variables on the integers $\{1, 2, \ldots, \theta\}$ where $\theta \geq 1$ is unknown. Let $T = \max\{X_1, X_2\}$.

(a) Show that T is sufficient for θ.

(b) Show that

$$P_\theta(T = t) = \begin{cases} \dfrac{2t-1}{\theta^2}, & \text{for } t = 1, 2, \ldots, \theta, \\ 0, & \text{otherwise.} \end{cases}$$

(c) Show that the family of distributions of T is complete.

(d) Find the minimum variance unbiased estimator of θ. *Hint*: You may wish to use the fact that

$$\sum_{k=1}^{n} (3k^2 - 3k + 1) = n^3.$$

4. (*24 points*) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed Poisson random variables with mean $\lambda > 0$ so that their common density function is

$$P_\lambda(X_1 = x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

for $x = 0, 1, 2, \ldots$. Assume that $n \geq 2$.

(a) If $\theta = P_\lambda(X_1 \leq 1)$, determine $\hat{\theta}(X_1, \ldots, X_n)$, the maximum likelihood estimator of θ.

(b) Is $\hat{\theta}(X_1, \ldots, X_n)$ an unbiased estimator of θ?

(c) For $\lambda > 0$, determine the asymptotic distribution of

$$\sqrt{n}\,\left(\hat{\theta}(X_1, \ldots, X_n) - \theta\right)$$

as $n \to \infty$.

5. (*12 points*) Let $X_1, X_2, \ldots, X_n$ be jointly normally distributed with

$$E(X_i) = \theta, \quad \mathrm{Var}(X_i) = \sigma^2 \text{ where } \sigma > 0,$$

and

$$\mathrm{Cov}(X_i, X_j) = \sigma^2 \rho$$

for $i \neq j$ where $\rho > 0$. Assuming that σ and ρ are known and that $\theta \in \mathbb{R}$ is a parameter, show that

$$\frac{1}{n} \sum_{i=1}^{n} X_i$$

is not a consistent estimator of θ.

6. (*10 points*) Let $X_1, X_2, \ldots, X_n$ be a random sample from the uniform distribution on $(0, \theta)$ where $\theta > 0$ is a parameter. Let

$$T = \max_{1 \leq i \leq n} X_i$$

and let

$$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

be the sample mean of X_i. Show that

$$E\left(\overline{X} \,|\, T = t\right) = \frac{n+1}{2n} t.$$

Hint: Do **not** try to find the conditional distribution of $\overline{X}$ given $T = t$. Use the Rao-Blackwell theorem instead.