Lecture #5: The Borel Sets of $\mathbb{R}$

We will now begin investigating the second of the two claims made at the end of Lecture #3, namely that there exists a σ-algebra $\mathcal{B}_1$ of subsets of $[0,1]$ on which it is possible to define a uniform probability.

Our goal for today will be to define the Borel sets of $\mathbb{R}$. The actual construction of the uniform probability will be deferred for several lectures.

Recall that a set $E \subseteq \mathbb{R}$ is said to be *open* if for every $x \in E$ there exists some $\epsilon > 0$ (depending on x) such that the interval $(x-\epsilon, x+\epsilon)$ is contained in E. Also recall that intervals of the form (a,b) for $-\infty < a < b < \infty$ are open sets. A set $F \subseteq \mathbb{R}$ is said to be *closed* if F^c is open. Note that both $\mathbb{R}$ and $\emptyset$ are simultaneously both open and closed sets.

If we consider the collection $\mathcal{O}$ of all open sets of $\mathbb{R}$, then it follows immediately that $\mathcal{O}$ is not a σ-algebra of subsets of $\mathbb{R}$. (That is, if $A \in \mathcal{O}$ so that A is an open set, then, by definition, A^c is closed and so $A^c \notin \mathcal{O}$.) However, we know that $\sigma(\mathcal{O})$, the σ-algebra generated by $\mathcal{O}$, exists and satisfies $\mathcal{O} \subsetneq \sigma(\mathcal{O}) \subseteq 2^{\mathbb{R}}$. This leads to the following definition.

Definition. The *Borel σ-algebra* of $\mathbb{R}$, written $\mathcal{B}$, is the σ-algebra generated by the open sets. That is, if $\mathcal{O}$ denotes the collection of all open subsets of $\mathbb{R}$, then $\mathcal{B} = \sigma(\mathcal{O})$.

Since $\mathcal{B}$ is a σ-algebra, we see that it necessarily contains all open sets, all closed sets, all unions of open sets, all unions of closed sets, all intersections of closed sets, and all intersections of open sets.

Exercise 5.1. The purpose of this exercise is to is to remind you of some facts about open and closed sets. Suppose that $\{E_1, E_2, \ldots\}$ is an arbitrary collection of open subsets of $\mathbb{R}$, and suppose that $\{F_1, F_2, \ldots\}$ is an arbitrary collection of closed subsets of $\mathbb{R}$. Prove that

(a) $\bigcup_{j=1}^\infty E_j$ is necessarily open,

(b) $\bigcap_{j=1}^\infty E_j$ need not be open,

(c) $\bigcap_{j=1}^\infty F_j$ is necessarily closed, and

(d) $\bigcup_{j=1}^\infty F_j$ need not be closed.

We often call a countable intersection of open sets a G_δ set (from the German *Gebeit* for open and *Durchschnitt* for intersection) and a countable union of closed sets an F_σ set (from the French *fermé* for closed and *somme* for union).

The following theorem characterizes open subsets of $\mathbb{R}$ and will occasionally be of use.

Theorem 5.2. *If $E \subseteq \mathbb{R}$ is an open set, then there exist at most countably many disjoint open intervals I_j, $j = 1, 2, \ldots$, such that*

$$E = \bigcup_{j=1}^{\infty} I_j.$$

Proof. The trick is to define an equivalence relation on E as follows. If a, $b \in E$, we say that a and b are equivalent, written $a \sim b$, if the entire open interval (a, b) is contained in E. This equivalence relationship partitions E into a disjoint union of classes. We do not know *a priori* that there are at most countably many such classes. Therefore, label these classes I_j, $j \in J$, where J is an arbitrary index set. Note that I_j is, in fact, an interval for the following reason. If a_j, $b_j \in I_j$, then $a_j \sim b_j$ so that the entire open interval (a_j, b_j) is contained in I_j. As a_j, b_j were arbitrary we see that I_j is, in fact, an interval. The next step is to show that I_j is open. Let $x \in I_j$ be arbitrary. Since $x \in E$ and E is open, we know that there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq E$. However, we clearly have $a \sim x$ for every $a \in (x - \epsilon, x + \epsilon)$ which implies that this ϵ-neighbourhood of x is contained in I_j. Thus, I_j is open as required. The final set is to show that there are at most countably many I_j. This follows from the fact that each I_j must contain at least one rational number. Since there are countably many rationals, there can be at most countably many I_j. $\square$

It is worth noting that we have not yet proved that $\mathcal{B} \neq 2^{\mathbb{R}}$; in other words, we have not yet proved that there exist non-Borel sets. It turns out that the set H constructed in Lecture #4 is non-Borel, although we will not prove this at present. In fact, there is no simple procedure to determine whether a given set $A \subseteq \mathbb{R}$ is Borel or not. However, one way to understand $\mathcal{B}$ is that it is generated by intervals of the form $(-\infty, a]$ as the next theorem shows.

Theorem 5.3. *The Borel σ-algebra $\mathcal{B}$ is generated by intervals of the form $(-\infty, a]$ where $a \in \mathbb{Q}$ is a rational number.*

Proof. Let $\mathcal{O}_0$ denote the collection of all open intervals. Since every open set in $\mathbb{R}$ is an at most countable union of open intervals, we must have $\sigma(\mathcal{O}_0) = \mathcal{B}$. Let $\mathcal{D}$ denote the collection of all intervals of the form $(-\infty, a]$, $a \in \mathbb{Q}$. Let $(a, b) \in \mathcal{O}_0$ for some $b > a$ with a, $b \in \mathbb{Q}$. Let

$$a_n = a + \frac{1}{n}$$

so that $a_n \downarrow a$ as $n \to \infty$, and let

$$b_n = b - \frac{1}{n}$$

so that $b_n \uparrow b$ as $n \to \infty$. Thus,

$$(a, b) = \bigcup_{n=1}^{\infty} (a_n, b_n] = \bigcup_{n=1}^{\infty} \{(-\infty, b_n] \cap (-\infty, a_n]^c\}$$

which implies that $(a, b) \in \sigma(\mathcal{D})$. That is, $\mathcal{O}_0 \subseteq \sigma(\mathcal{D})$ so that $\sigma(\mathcal{O}_0) \subseteq \sigma(\mathcal{D})$. However, every element of $\mathcal{D}$ is a closed set which implies that

$$\sigma(\mathcal{D}) \subseteq \mathcal{B}.$$

This gives the chain of containments

$$\mathcal{B} = \sigma(\mathcal{O}_0) \subseteq \sigma(\mathcal{D}) \subseteq \mathcal{B}$$

and so $\sigma(\mathcal{D}) = \mathcal{B}$ proving the theorem. $\square$

The Borel sets of $[0, 1]$

If we now consider the set $[0, 1] \subset \mathbb{R}$ as the sample space, then $\mathcal{B}_1$, the Borel σ-algebra of $[0, 1]$, is the σ-algebra generated by the collection of open subsets of $[0, 1]$. As the next exercise shows, we can equivalently think of $\mathcal{B}_1$ as the restriction of $\mathcal{B}$ to $[0, 1]$. Note that $\mathcal{B}_1 \subset \mathcal{B}$ as collections of sets, but not as σ-algebras. That is, $\mathcal{B}_1$ is not a sub-σ-algebra of $\mathcal{B}$. The reason, of course, is that $\mathcal{B}$ is a σ-algebra of subsets of $\mathbb{R}$ whereas $\mathcal{B}_1$ is a σ-algebra of subsets of $[0, 1]$; in order for one σ-algebra to be a sub-σ-algebra of another σ-algebra, it is necessarily the case that the underlying sample spaces for both σ-algebras are the same.

Exercise 5.4. Suppose that Ω is the sample space and $\Omega' \subseteq \Omega$. Show that

(a) if $\mathcal{F}$ is a σ-algebra of subsets of Ω and $\mathcal{F}' = \{A \cap \Omega' : A \in \mathcal{F}\}$, then $\mathcal{F}'$ is a σ-algebra of subsets of Ω', and

(b) if $\mathcal{C}$ generates the σ-algebra $\mathcal{F}$ in Ω and $\mathcal{C}' = \{A \cap \Omega' : A \in \mathcal{C}\}$, then $\mathcal{C}'$ generates the σ-algebra $\mathcal{F}'$ in Ω'.