Statistics 451 (Fall 2013)
Some More Basic Set Theory

Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function.The range of f is

$$\{y \in \mathbb{R}^m : f(x) = y \text{ for some } x \in \mathbb{R}^n\}.$$

Note that the range of f is a subset of $\mathbb{R}^m$. If $A \subseteq \mathbb{R}^n$, then the image of A under f is the subset of the range (and therefore a subset of $\mathbb{R}^m$) given by

$$f(A) = \{f(x) : x \in A\}.$$

If $B \subseteq \mathbb{R}^m$, then the preimage of B under f is the subset of the domain of f (and therefore a subset of $\mathbb{R}^n$) given by

$$f^{-1}(B) = \{x \in A : f(x) = y \text{ for some } y \in B\}.$$

Example. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^2$. Determine

(i) the range of f,

(ii) the image of $A = [-2, 2]$ under f,

(iii) the image of $A = (-5, -3) \cup [1, 4]$ under f, and

(iv) the preimage of $B = (9, 25)$ under f.

Solution.

(i) The range of f is $[0, \infty)$.

(ii) The image of $[-2, 2]$ is $f([-2, 2]) = [0, 4]$.

(iii) The image of $(-5, -3) \cup [1, 4]$ is $f((-5, -3) \cup [1, 4]) = (9, 25) \cup [1, 16] = [1, 25)$.

(iv) The preimage of $(9, 25)$ is $f^{-1}((9, 25)) = (-5, -3) \cup (3, 5)$.

Exercise. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is *any* function. Show that

(i) if $B \subseteq \mathbb{R}^m$, then $[f^{-1}(B)]^c = f^{-1}(B^c)$,

(ii) if $B_1, B_2, \ldots \subseteq \mathbb{R}^m$, then

$$f^{-1}\left(\bigcup_{j=1}^{\infty} B_j\right) = \bigcup_{j=1}^{\infty} f^{-1}(B_j),$$

and

(iii) if $B_1, B_2, \ldots \subseteq \mathbb{R}^m$, then

$$f^{-1}\left(\bigcap_{j=1}^{\infty} B_j\right) = \bigcap_{j=1}^{\infty} f^{-1}(B_j).$$

In particular, it follows from (iii) that if $B_1, B_2, \ldots \subseteq \mathbb{R}^m$ are disjoint, then $f^{-1}(B_1), f^{-1}(B_2), \ldots$ are disjoint.