Stat 441 Winter 2009
Assignment #6

This assignment is due at the beginning of class on Friday, February 27, 2009.

1. Suppose that $\{B_t, t \geq 0\}$ is a standard Brownian motion with $B_0 = 0$. Determine an expression for

$$\int_0^t \sin(B_s)\, \mathrm{d}B_s$$

that does not involve Itô integrals.

2. Suppose that $\{B_t, t \geq 0\}$ is a standard Brownian motion with $B_0 = 0$, and suppose further that the process $\{X_t, t \geq 0\}$, $X_0 = a > 0$, satisfies the stochastic differential equation

$$\mathrm{d}X_t = X_t\, \mathrm{d}B_t + \frac{1}{X_t}\, \mathrm{d}t.$$

(a) If $f(x) = x^2$, determine $\mathrm{d}f(X_t)$.

(b) If $f(t, x) = t^2x^2$, determine $\mathrm{d}f(t, X_t)$.

3. We know from Theorem 14.6 that any Itô integral is a martingale. If we combine this fact with Itô's formula, then we have a method for "generating" martingales. That is, if we can find functions f for which we can make the $\mathrm{d}t$ term in Itô's formula vanish, then we have found a martingale. For instance, Version I of Itô's formula tells us that

$$\mathrm{d}f(B_t) = f'(B_t)\, \mathrm{d}B_t + \frac{1}{2}f''(B_t)\, \mathrm{d}t.$$

Hence, if we can find $f(x)$ such that $f''(x) = 0$, then $f(B_t)$ will be a martingale. Since $f''(x) = 0$ implies that $f(x) = ax + b$ where a, $b \in \mathbb{R}$ are arbitrary constants, any linear transformation of Brownian motion is a martingale. That is, $\{M_t, t \geq 0\}$ where $M_t = aB_t + b$ is a martingale.

More interesting examples arise when we consider Version II of Itô's formula which tells us that

$$\mathrm{d}f(t, B_t) = f'(t, B_t)\, \mathrm{d}B_t + \left[\dot{f}(t, B_t) + \frac{1}{2}f''(t, B_t)\right]\, \mathrm{d}t.$$

Hence, if we can find $f(t, x)$ such that

$$\dot{f}(t, x) + \frac{1}{2}f''(t, x) = 0,$$

then $f(t, B_t)$ will be a martingale.

Notice that $f(t, x) = x^2 - t$, $f(t, x) = x^3 - 3tx$, and $f(t, x) = x^4 - 6tx^2 + 3t^2$ all work. (Take a look back at Exercise 7.6; you can now see how I created that exercise.)

(a) Find functions (of the two variables t and x) that contain leading terms x^5 and x^6, respectively, that generate martingales.

There are, in fact, non-polynomial solutions to this equation such as

$$f(t,x) = e^{t/2}\sin(x).$$

(b) Find some other non-polynomial solutions, including one involving $\cos(x)$.

4. Suppose that $\{B_t, t \ge 0\}$ is a standard Brownian motion, and let $\{\mathcal{F}_t, t \ge 0\}$ denote the Brownian filtration. Problem #4 on Assignment #3 asked you to compute $\mathbb{E}(\sin(B_t)|\mathcal{F}_s)$ for $0 \le s < t$ and to use this result to find a function of $\sin(B_t)$ that is a martingale. The reason that the problem was too difficult for us to solve at the time was that we were unable to evaluate the resulting integral. That is, suppose that $s < t$ so that the addition formula for sine implies

$$\sin(B_t) = \sin(B_t - B_s + B_s) = \sin(B_t - B_s)\cos(B_s) + \sin(B_s)\cos(B_t - B_s).$$

Thus,

$$\mathbb{E}(\sin(B_t)|\mathcal{F}_s) = \cos(B_s)\mathbb{E}[\sin(B_t - B_s)] + \sin(B_s)\mathbb{E}[\cos(B_t - B_s)]$$

using the independence of Brownian increments and properties of conditional expectation. Since $B_t - B_s \sim \mathcal{N}(0, t-s)$, we can write

$$\mathbb{E}[\sin(B_t - B_s)] = \frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^{\infty} \exp\left\{-\frac{x^2}{2(t-s)}\right\}\sin(x)\,\mathrm{d}x$$

and

$$\mathbb{E}[\cos(B_t - B_s)] = \frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^{\infty} \exp\left\{-\frac{x^2}{2(t-s)}\right\}\cos(x)\,\mathrm{d}x.$$

The fact that $e^{-x^2}\sin(x)$ is an odd function implies that $\mathbb{E}[\sin(B_t - B_s)] = 0$. The fact that $e^{-x^2}\cos(x)$ is an even function implies that

$$\mathbb{E}[\cos(B_t - B_s)] = \frac{2}{\sqrt{2\pi(t-s)}}\int_{0}^{\infty} \exp\left\{-\frac{x^2}{2(t-s)}\right\}\cos(x)\,\mathrm{d}x.$$

Hence, we find

$$\mathbb{E}(\sin(B_t)|\mathcal{F}_s) = \left[\frac{2}{\sqrt{2\pi(t-s)}}\int_{0}^{\infty} \exp\left\{-\frac{x^2}{2(t-s)}\right\}\cos(x)\,\mathrm{d}x\right]\sin(B_s). \qquad (*)$$

The previous problem implies that if $M_t = e^{t/2}\sin(B_t)$, then $\{M_t, t \ge 0\}$ is a martingale with respect to the Brownian filtration. This means that $\mathbb{E}(M_t|\mathcal{F}_s) = M_s$, or equivalently,

$$\mathbb{E}(e^{t/2}\sin(B_t)|\mathcal{F}_s) = e^{s/2}\sin(B_s)$$

so that

$$\mathbb{E}(\sin(B_t)|\mathcal{F}_s) = e^{-(t-s)/2}\sin(B_s). \qquad (**)$$

Equating $(*)$ and $(**)$ therefore implies that

$$\frac{2}{\sqrt{2\pi(t-s)}}\int_{0}^{\infty} \exp\left\{-\frac{x^2}{2(t-s)}\right\}\cos(x)\,\mathrm{d}x = e^{-(t-s)/2}.$$

Using **(b)** of the previous exercise, mimic this calculation and compute $\mathbb{E}(\cos(B_t)|\mathcal{F}_s)$.

The value of this integral can also be found directly using the theory of residues as taught in Math 312: Complex Analysis.