Stat 441 Winter 2009
Assignment #3

This assignment is due at the beginning of class on Wednesday, January 28, 2009.

1. Read Chapters 6 and 7, pages 53–72, of Higham.

- Chapters 6 and 7 develop "geometric Brownian motion" as the model of the stock price movement. Read through these chapters to get a feel for what is to come.
- Complete the MATLAB exercises from Chapters 6 and 7.

2. (VERY EASY) Suppose that $\{X_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$ are both continuous-time martingales with respect to the filtration $\{\mathcal{F}_t, t \geq 0\}$. Show that the process $\{Z_t, t \geq 0\}$ defined by $Z_t = X_t + Y_t$ is a continuous-time martingale with respect to the filtration $\{\mathcal{F}_t, t \geq 0\}$.

3. The following exercises are from the printed lecture notes.

- Exercises 7.1, 7.4, 7.5, and 7.6

4. Suppose that $\{B_t, t \geq 0\}$ is a standard Brownian motion starting at 0, and let $\{F_t, t \geq 0\}$ denote the Brownian filtration. Let $s < t$ and compute $\mathbb{E}(\sin(B_t)|\mathcal{F}_s)$. Is $\{\sin(B_t), t \geq 0\}$ a martingale with respect to $\{\mathcal{F}_t, t \geq 0\}$. If not, find a function of $\sin(B_t)$ that is a martingale.

5. Let $\{B_t, t \geq 0\}$ be a Brownian motion which, as we know, is a continuous-time martingale with respect to the Brownian filtration $\{\mathcal{F}_t, t \geq 0\}$. Let $t_0 = 0 < t_1 < t_2 < \cdots$ be any countable subset of $[0, \infty)$ and consider $\{B_{t_j}, j = 0, 1, 2, \ldots\}$. Verify that $\{B_{t_j}, j = 0, 1, 2, \ldots\}$ is a discrete-time martingale with respect to the filtration $\{\mathcal{F}_{t_j}, j = 0, 1, 2, \ldots\}$. Notationally, you might find it helpful to define $\{X_j, j = 0, 1, 2, \ldots\}$ by setting $X_j = B_{t_j}$ and show that $\{X_j, j = 0, 1, 2, \ldots\}$ is a discrete-time martingale with respect to the filtration $\{\mathcal{G}_j, j = 0, 1, 2, \ldots\}$ where $\mathcal{G}_j = \mathcal{F}_{t_j}$.

6. Suppose that the price of a stock $\{X_t, t \geq 0\}$ follows a Brownian motion. If the price of the stock at time 3 is \$52, determine the probability that the price of the stock is at least \$55 at time 12.

7. The exchange rate of yen per dollar follows a Brownian motion. The following exchange rates are given.

Day	¥/\$
1	105
2	104
3	102

Calculate the probability that the exchange rate is between 102 and 105 on day 5.

8. If $Z \sim \mathcal{N}(0, 1)$, then the process $\{X_t, t \geq 0\}$ defined by setting $X_t = \sqrt{t}\,Z$ is continuous and is marginally distributed as $\mathcal{N}(0, t)$. Is $\{X_t, t \geq 0\}$ a Brownian motion starting at 0?

(continued)

9. Suppose that $\{B_t, t \geq 0\}$ and $\{\tilde{B}_t, t \geq 0\}$ are two independent Brownian motions starting at 0. Let α be a constant with $-1 < \alpha < 1$, and define the process $\{X_t, t \geq 0\}$ by setting $X_t = \alpha B_t + \sqrt{1-\alpha^2}\,\tilde{B}_t$. It follows that this process is continuous and is marginally distributed as $\mathcal{N}(0,t)$. Is $\{X_t, t \geq 0\}$ a Brownian motion starting at 0?

10. Suppose that $\{B_t, t \geq 0\}$ is a Brownian motion starting at 0. Let μ and $\sigma > 0$ be real constants. Define the process $\{X_t, t \geq 0\}$ by setting $X_t = \sigma B_t + \mu t$. If $T > 0$, show that $\mathbf{P}\{X_T < 0\} > 0$. *Hint: What is the (marginal) distribution of* X_T*?*