Lecture #7, 8: Brownian Motion as a Model of a Fair Game

Suppose that we are interested in setting up a model of a fair game, and that we are going to place bets on the outcomes of the individual rounds of this game. If we assume that a round takes place at discrete times, say at times $1, 2, 3, \ldots$, and that the game pays even money on unit stakes per round, then a reasonable probability model for encoding the outcome of the jth game is via a sequence $\{X_j, j = 1, 2, \ldots\}$ of independent and identically distributed random variables with

$$\mathbf{P}\{X_1 = 1\} = \mathbf{P}\{X_1 = -1\} = \frac{1}{2}.$$

That is, we can view X_j as the outcome of the jth round of this fair game. Although we will assume that there is no game played at time 0, it will be necessary for our notation to consider what "happens" at time 0; therefore, we will simply define $X_0 = 0$.

Notice that the sequence $\{X_j, j = 1, 2, \ldots\}$ tracks the outcomes of the individual games. We would also like to track our net number of "wins"; that is, we care about

$$\sum_{j=1}^{n} X_j,$$

the net number of "wins" after n rounds. (If this sum is negative, we realize that a negative number of "wins" is an interpretation of a net "loss.") Hence, we define the process $\{S_n, n = 0, 1, 2, \ldots\}$ by setting

$$S_n = \sum_{j=0}^{n} X_j.$$

Of course, we know that $\{S_n, n = 0, 1, 2, \ldots\}$ is called a *simple random walk*, and so we use a simple random walk as our model of a fair game being played in discrete time.

If we write $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$ to denote the *information* contained in the first n rounds of this game, then we know from our earlier work that $\{S_n, n = 0, 1, 2, \ldots\}$ is a martingale with respect to the filtration $\{\mathcal{F}_n, n = 0, 1, 2, \ldots\}$.

Notice that $S_j - S_{j-1} = X_j$ and so the *increment* $S_j - S_{j-1}$ is exactly the outcome of the jth round of this fair game.

Suppose that we bet on the outcome of the jth round of this game and that (as assumed above) the game pays even money on unit stakes; for example, if we flip a fair coin betting \$5 on "heads" and "heads" does, in fact, appear, then we win \$5 plus our original \$5, but if "tails" appears, then we lose our original \$5.

If we denote our betting strategy by Y_{j-1}, $j = 1, 2, \ldots$, so that Y_{j-1} represents the bet we make on the jth round of the game, then I_n, our *fortune* after n rounds, is given by

$$I_n = \sum_{j=1}^{n} Y_{j-1}(S_j - S_{j-1}). \tag{5.1}$$

We also *define* $I_0 = 0$. The process $\{I_n, n = 0, 1, 2, \ldots\}$ is called a *discrete stochastic integral* (or the *martingale transform of* Y *by* S).

Remark. If we choose unit bets each round so that $Y_{j-1} = 1$, $j = 1, 2, \ldots$, then

$$I_n = \sum_{j=1}^{n} (S_j - S_{j-1}) = S_n$$

and so our "fortune" after n rounds is simply the position of the random walk S_n. We are interested in what happens when Y_{j-1} is *not* constant in time, but rather varies with j.

Note that it is reasonable to assume that the bet you make on the jth round can only depend on the outcomes of the previous $j-1$ rounds. That is, you cannot "look into the future and make your bet on the jth round based on what the outcome of the jth round will be." In mathematical language, we say that Y_{j-1} must be *previsible* (also called *predictable*).

Remark. The concept of a *previsible stochastic process* was intensely studied in the 1950s by the French school of probability that included P. Lévy. Since the French word *prévisible* is translated into English as *foreseeable*, there is no consistent English translation. Most probabilists use previsible and predictable interchangeably. (Although, unfortunately, not all do!)

A slight modification of Example 4.9 shows that $\{I_n, n = 0, 1, 2, \ldots\}$ is a martingale with respect to the filtration $\{\mathcal{F}_n, n = 0, 1, \ldots\}$. Note that the requirement that Y_{j-1} be previsible is exactly the requirement that allows $\{I_n, n = 0, 1, 2, \ldots\}$ to be a martingale.

It now follows from Theorem 4.4 that $\mathbb{E}(I_n) = 0$ for all n since $\{I_n, n = 0, 1, 2, \ldots\}$ is a martingale with $I_0 = 0$. As we saw in Exercise 4.10, calculating the variance of the random variable I_n is more involved. The following exercise generalizes that result and shows precisely how the variance depends on the choice of the sequence Y_{j-1}, $j = 1, 2, \ldots$.

Exercise 5.1. Consider the martingale transform of Y by S given by (5.1). Show that

$$\operatorname{Var}(I_n) = \sum_{j=1}^{n} \mathbb{E}(Y_{j-1}^2).$$

Suppose that instead of playing a round of the game at times $1, 2, 3, \ldots$, we play rounds more frequently, say at times $0.5, 1, 1.5, 2, 2.5, 3, \ldots$, or even more frequently still. In fact, we can imagine playing a round of the game at *every* time $t \geq 0$.

If this is hard to visualize, imagine the round of the game as being the price of a (fair) stock at time t. The stock is assumed, equally likely, to move an infinitesmal amount up or an infinitesmal amount down in every infinitesmal period of time.

Hence, if we want to model a fair game occurring in continuous time, then we need to find a *continuous limit* of the simple random walk. This continuous limit is Brownian motion, also called the *scaling limit* of simple random walk. To explain what this means, suppose that $\{S_n, n = 0, 1, 2, \ldots\}$ is a simple random walk. For $N = 1, 2, 3, \ldots$, define the scaled random walk $B_t^{(N)}$, $0 \le t \le 1$, to be the continuous process on the time interval $[0,1]$ whose value at the fractional times $0, \frac{1}{N}, \frac{2}{N}, \dots, \frac{N-1}{N}, 1$ is given by setting

$$B_{\frac{j}{N}}^{(N)} = \frac{1}{\sqrt{N}} S_j, \quad j = 0, 1, 2, \dots, N,$$

and for other times is defined by linear interpolation. As $N \to \infty$, the distribution of the process $\{B_t^{(N)}, 0 \le t \le 1\}$ converges to the distribution of a process $\{B_t, 0 \le t \le 1\}$ satisfying the following properties:

- $B_0 = 0$,
- for any $0 \le s \le t \le 1$, the random variable $B_t - B_s$ is normally distributed with mean 0 and variance $t - s$; that is, $B_t - B_s \sim \mathcal{N}(0, t-s)$,
- for any integer k and any partition $0 \le t_1 \le t_2 \le \cdots \le t_k \le 1$, the random variables $B_{t_k} - B_{t_{k-1}}, \dots, B_{t_2} - B_{t_1}, B_{t_1}$ are independent, and
- the trajectory $t \mapsto B_t$ is continuous.

By piecing together independent copies of this process, we can construct a Brownian motion $\{B_t, t \ge 0\}$ defined for all times $t \ge 0$ satisfying the above properties (without, of course, the restriction in (b) that $t \le 1$ and the restriction in (c) that $t_k \le 1$). Thus, we now suppose that $\{B_t, t \ge 0\}$ is a Brownian motion with $B_0 = 0$.

Exercise 5.2. Deduce from the definition of Brownian motion that for each $t > 0$, the random variable B_t is normally distributed with mean 0 and variance t. Why does this imply that $\mathbb{E}(B_t^2) = t$?

Exercise 5.3. Deduce from the definition of Brownian motion that for $0 \le s < t$, the distribution of the random variable $B_t - B_s$ is the *same* as the distribution of the random variable B_{t-s}.

Exercise 5.4. Show that if $\{B_t, t \ge 0\}$ is a Brownian motion, then $\mathbb{E}(B_t) = 0$ for all t, and $\mathrm{Cov}(B_s, B_t) = \min\{s, t\}$. *Hint: Suppose that $s < t$ and write $B_s B_t = (B_s B_t - B_s^2) + B_s^2$.* The result of this exercise actually shows that Brownian motion is not a stationary process, although it does have stationary increments.

Note. One of the problems with using either simple random walk or Brownian motion as a model of an asset price is that the value of a real stock is never allowed to be negative—it can equal 0, but can never be strictly less than 0. On the other hand, both a random walk and a Brownian motion can be negative. Hence, neither serves as an adequate model for a stock. Nonetheless, Brownian motion is the key ingredient for building a reasonable model of a stock and the stochastic integral that we are about to construct is *fundamental* to the analysis. At this point, we must be content with modelling (and betting on) fair games whose values can be either positive or negative.

If we let $\mathcal{F}_t = \sigma(B_s, 0 \le s \le t)$ denote the "information" contained in the Brownian motion up to (and including) time t, then it easily follows that $\{B_t, t \ge 0\}$ is a continuous-time martingale with respect to the Brownian filtration $\{\mathcal{F}_t, t \ge 0\}$. That is, suppose that $s < t$, and so

$$\mathbb{E}(B_t|\mathcal{F}_s) = \mathbb{E}(B_t - B_s + B_s|\mathcal{F}_s) = \mathbb{E}(B_t - B_s|\mathcal{F}_s) + \mathbb{E}(B_s|\mathcal{F}_s) = \mathbb{E}(B_t - B_s) + B_s = B_s$$

since the Brownian increment $B_t - B_s$ has mean 0 and is independent of $\mathcal{F}_s$, and B_s is "known" at time s (using the "taking out what is known" property of conditional expectation).

In analogy with simple random walk, we see that although $\{B_t^2, t \ge 0\}$ is not a martingale with respect to $\{\mathcal{F}_t, t \ge 0\}$, the process $\{B_t^2 - t, t \ge 0\}$ *is* one.

Exercise 5.5. Let the process $\{M_t, t \ge 0\}$ be defined by setting $M_t = B_t^2 - t$. Show that $\{M_t, t \ge 0\}$ is a (continuous-time) martingale with respect to the Brownian filtration $\{\mathcal{F}_t, t \ge 0\}$.

Exercise 5.6. The same "trick" used to solve the previous exercise can also be used to show that both $\{B_t^3 - 3tB_t, t \ge 0\}$ and $\{B_t^4 - 6tB_t^2 + 3t^2, t \ge 0\}$ are martingales with respect to the Brownian filtration $\{\mathcal{F}_t, t \ge 0\}$. Verify that these are both, in fact, martingales. (Once we have learned Itô's formula, we will discover a much easier way to "generate" such martingales.)

Assuming that our fair game is modelled by a Brownian motion, we need to consider appropriate betting strategies. For now, we will allow only deterministic betting strategies that do not "look into the future" and denote such a strategy by $\{g(t), t \ge 0\}$. This notation might look a little strange, but it is meant to be suggestive for when we allow certain random betting strategies. Hence, at this point, our betting strategy is simply a real-valued function $g : [0, \infty) \to \mathbb{R}$. Shortly, for technical reasons, we will see that it is necessary for g to be at least bounded, piecewise continuous, and in $L^2([0, \infty))$. Recall that $g \in L^2([0, \infty))$ means that

$$\int_0^\infty g^2(s)\, \mathrm{d}s < \infty.$$

Thus, if we fix a time $t > 0$, then, in analogy with (5.1), our "fortune process" up to time t is given by the (yet-to-be-defined) *stochastic integral*

$$I_t = \int_0^t g(s)\, \mathrm{d}B_s. \tag{5.2}$$

Our goal, now, is to try and define (5.2) in a reasonable way. A natural approach, therefore, is to try and relate the stochastic integral (5.2) with the discrete stochastic integral (5.1) constructed earlier. Since the discrete stochastic integral resembles a Riemann sum, that seems like a good place to start.