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Prof. Michael Kozdron

Lecture #4, 5, 6: Discrete-Time Martingales

The concept of a martingale is fundamental to modern probability and is one of the key tools needed to study mathematical finance. Although we saw the definition in STAT 351, we are now going to need to be a little more careful than we were in that class. This will be especially true when we study continuous-time martingales.

Definition 4.1. A sequence $X_0, X_1, X_2, \ldots$ of random variables is said to be a *martingale* if

$$\mathbb{E}(X_{n+1}|X_0, X_1, \ldots, X_n) = X_n$$

for every $n = 0, 1, 2, \ldots$.

Technically, we need all of the random variables to have finite expectation in order that conditional expectations be defined. Furthermore, we will find it useful to introduce the following notation. Let $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$ denote the *information* contained in the sequence $\{X_0, X_1, \ldots, X_n\}$ up to (and including) time n. We then call the sequence $\{\mathcal{F}_n, n = 0, 1, 2, \ldots\} = \{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots\}$ a *filtration.*

Definition 4.2. A sequence $\{X_n, n = 0, 1, 2 \ldots\}$ of random variables is said to be a *martingale* with respect to the filtration $\{\mathcal{F}_n, n = 0, 1, 2, \ldots\}$ if

(i) $X_n \in \mathcal{F}_n$ for every $n = 0, 1, 2, \ldots$,

(ii) $\mathbb{E}|X_n| < \infty$ for every $n = 0, 1, 2, \ldots$, and

(iii) $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$ for every $n = 0, 1, 2, \ldots$.

If $X_n \in \mathcal{F}_n$, then we often say that X_n is *adapted.* The intuitive idea is that if X_n is adapted, then X_n is "known" at time n. In fact, you are already familiar with this notion from STAT 351.

Remark. Suppose that n is fixed, and let $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$. Clearly $\mathcal{F}_{n-1} \subset \mathcal{F}_n$ and so $X_1 \in \mathcal{F}_n$, $X_2, \in \mathcal{F}_n, \ldots, X_n \in \mathcal{F}_n$.

Moreover, the following theorem is extremely useful to know when working with martingales.

Theorem 4.3. *Let $X_1, X_2, \ldots, X_n$, Y be random variables, let $g : \mathbb{R}^n \to \mathbb{R}$ be a function, and let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. It then follows that*

- $\mathbb{E}(g(X_1, X_2, \ldots, X_n)\, Y|\mathcal{F}_n) = g(X_1, X_2, \ldots, X_n)\mathbb{E}(Y|\mathcal{F}_n)$ *(taking out what is known),*
- $\mathbb{E}(Y|\mathcal{F}_n) = \mathbb{E}(Y)$ *if Y is independent of $\mathcal{F}_n$, and*
- $\mathbb{E}(\mathbb{E}(Y|\mathcal{F}_n)) = \mathbb{E}(Y)$.

One useful fact about martingales is that they have stable expectation.

Theorem 4.4. *If* $\{X_n, n = 0, 1, 2, \ldots\}$ *is a martingale, then* $\mathbb{E}(X_n) = \mathbb{E}(X_0)$ *for every* $n = 0, 1, 2, \ldots$.

Proof. Since

$$\mathbb{E}(X_{n+1}) = \mathbb{E}(\mathbb{E}(X_{n+1}|\mathcal{F}_n)) = \mathbb{E}(X_n),$$

we can iterate to conclude that

$$\mathbb{E}(X_{n+1}) = \mathbb{E}(X_n) = \mathbb{E}(X_{n-1}) = \cdots = \mathbb{E}(X_0)$$

as required. □

Exercise 4.5. Suppose that $\{X_n, n = 1, 2, \ldots\}$ is a discrete-time stochastic process. Show that $\{X_n, n = 1, 2, \ldots\}$ is a martingale with respect to the filtration $\{\mathcal{F}_n, n = 0, 1, 2, \ldots\}$ if and only if

(i) $X_n \in \mathcal{F}_n$ for every $n = 0, 1, 2, \ldots$,

(ii) $\mathbb{E}|X_n| < \infty$ for every $n = 0, 1, 2, \ldots$, and

(iii) $\mathbb{E}(X_n|\mathcal{F}_m) = X_m$ for every integer m with $0 \le m < n$.

We are now going to study several examples of martingales. Most of them are variants of simple random walk which we define in the next example.

Example 4.6. Suppose that $Y_1, Y_2, \ldots$ are independent, identically distributed random variables with $\mathbf{P}\{Y_1 = 1\} = \mathbf{P}\{Y = -1\} = 1/2$. Let $S_0 = 0$, and for $n = 1, 2, \ldots$, define $S_n = Y_1 + Y_2 + \cdots + Y_n$. The sequence $\{S_n, n = 0, 1, 2, \ldots\}$ is called a *simple random walk (starting at* 0*)*. Before we show that $\{S_n, n = 0, 1, 2, \ldots\}$ is a martingale, it will be useful to calculate $\mathbb{E}(S_n)$, $\text{Var}(S_n)$, and $\text{Cov}(S_n, S_{n+1})$. Observe that

$$(Y_1 + Y_2 + \cdots + Y_n)^2 = Y_1^2 + Y_2^2 + \cdots + Y_n^2 + \sum_{i \ne j} Y_i Y_j.$$

Since $\mathbb{E}(Y_1) = 0$ and $\text{Var}(Y_1) = \mathbb{E}(Y_1^2) = 1$, we find

$$\mathbb{E}(S_n) = \mathbb{E}(Y_1 + Y_2 + \cdots + Y_n) = \mathbb{E}(Y_1) + \mathbb{E}(Y_2) + \cdots + \mathbb{E}(Y_n) = 0$$

and

$$\begin{aligned}\text{Var}(S_n) = \mathbb{E}(S_n^2) = \mathbb{E}(Y_1 + Y_2 + \cdots + Y_n)^2 &= \mathbb{E}(Y_1^2) + \mathbb{E}(Y_2^2) + \cdots + \mathbb{E}(Y_n^2) + \sum_{i \ne j} \mathbb{E}(Y_i Y_j) \\ &= 1 + 1 + \cdots + 1 + 0 \\ &= n\end{aligned}$$

since $\mathbb{E}(Y_i Y_j) = \mathbb{E}(Y_i)\mathbb{E}(Y_j)$ when $i \ne j$ because of the assumed independence of $Y_1, Y_2, \ldots$. Since $S_{n+1} = S_n + Y_{n+1}$ we see that

$$\text{Cov}(S_n, S_{n+1}) = \text{Cov}(S_n, S_n + Y_{n+1}) = \text{Cov}(S_n, S_n) + \text{Cov}(S_n, Y_{n+1}) = \text{Var}(S_n) + 0$$

using the fact that Y_{n+1} is independent of S_n. Furthermore, since $\text{Var}(S_n) = n$, we conclude $\text{Cov}(S_n, S_{n+1}) = n$.

Exercise 4.7. As a generalization of this covariance calculation, show that $\mathrm{Cov}(S_n, S_m) = \min\{n, m\}$.

Example 4.6 (continued). We now show that the simple random walk $\{S_n, n = 0, 1, 2, \ldots\}$ is a martingale. This also illustrates the usefulness of the $\mathcal{F}_n$ notation since

$$\mathcal{F}_n = \sigma(S_0, S_1, \ldots, S_n) = \sigma(Y_1, \ldots, Y_n).$$

Notice that

$$\mathbb{E}(S_{n+1}|\mathcal{F}_n) = \mathbb{E}(Y_{n+1} + S_n|\mathcal{F}_n) = \mathbb{E}(Y_{n+1}|\mathcal{F}_n) + \mathbb{E}(S_n|\mathcal{F}_n).$$

Since Y_{n+1} is independent of $\mathcal{F}_n$ we conclude that

$$\mathbb{E}(Y_{n+1}|\mathcal{F}_n) = \mathbb{E}(Y_{n+1}) = 0.$$

If we condition on $\mathcal{F}_n$, then S_n is *known*, and so

$$\mathbb{E}(S_n|\mathcal{F}_n) = S_n.$$

Combined we conclude

$$\mathbb{E}(S_{n+1}|\mathcal{F}_n) = \mathbb{E}(Y_{n+1}|\mathcal{F}_n) + \mathbb{E}(S_n|\mathcal{F}_n) = 0 + S_n = S_n$$

which proves that $\{S_n, n = 0, 1, 2, \ldots\}$ is a martingale.

Example 4.6 (continued). Next we show that $\{S_n^2 - n, n = 0, 1, 2, \ldots\}$ is also a martingale. Let $M_n = S_n^2 - n$. We must show that $\mathbb{E}(M_{n+1}|\mathcal{F}_n) = M_n$ since

$$\mathcal{F}_n = \sigma(M_0, M_1, \ldots, M_n) = \sigma(S_0, S_1, \ldots, S_n).$$

Notice that

$$\mathbb{E}(S_{n+1}^2|\mathcal{F}_n) = \mathbb{E}((Y_{n+1} + S_n)^2|\mathcal{F}_n) = \mathbb{E}(Y_{n+1}^2|\mathcal{F}_n) + 2\mathbb{E}(Y_{n+1}S_n|\mathcal{F}_n) + \mathbb{E}(S_n^2|\mathcal{F}_n).$$

However,

- $\mathbb{E}(Y_{n+1}^2|\mathcal{F}_n) = \mathbb{E}(Y_{n+1}^2) = 1$,
- $\mathbb{E}(Y_{n+1}S_n|\mathcal{F}_n) = S_n\mathbb{E}(Y_{n+1}|\mathcal{F}_n) = S_n\mathbb{E}(Y_{n+1}) = 0$, and
- $\mathbb{E}(S_n^2|\mathcal{F}_n) = S_n^2$

from which we conclude that

$$\mathbb{E}(S_{n+1}^2|\mathcal{F}_n) = S_n^2 + 1.$$

Therefore,

$$\begin{aligned}\mathbb{E}(M_{n+1}|\mathcal{F}_n) = \mathbb{E}(S_{n+1}^2 - (n+1)|\mathcal{F}_n) = \mathbb{E}(S_{n+1}^2|\mathcal{F}_n) - (n+1) &= S_n^2 + 1 - (n+1) \\ &= S_n^2 - n \\ &= M_n\end{aligned}$$

and so we conclude that $\{M_n, n = 0, 1, 2, \ldots\} = \{S_n^2 - n, n = 0, 1, 2, \ldots\}$ is a martingale.

Example 4.6 (continued). We are now going to construct one more martingale related to simple random walk. Suppose that $\theta \in \mathbb{R}$ and let

$$Z_n = (\operatorname{sech}\theta)^n e^{\theta S_n}, \quad n = 0, 1, 2, \ldots,$$

where the hyperbolic secant is defined as

$$\operatorname{sech}\theta = \frac{2}{e^\theta + e^{-\theta}}.$$

We will show that $\{Z_n, n = 0, 1, 2, \ldots\}$ is a martingale. Thus, we must verify that

$$\mathbb{E}(Z_{n+1}|\mathcal{F}_n) = Z_n$$

since

$$\mathcal{F}_n = \sigma(Z_0, Z_1, \ldots, Z_n) = \sigma(S_0, S_1, \ldots, S_n).$$

Notice that $S_{n+1} = S_n + Y_{n+1}$ which implies

$$\begin{aligned} Z_{n+1} = (\operatorname{sech}\theta)^{n+1} e^{\theta S_{n+1}} = (\operatorname{sech}\theta)^{n+1} e^{\theta(S_n + Y_{n+1})} &= (\operatorname{sech}\theta)^n e^{\theta S_n} \cdot (\operatorname{sech}\theta) e^{\theta Y_{n+1}} \\ &= Z_n \cdot (\operatorname{sech}\theta) e^{\theta Y_{n+1}}. \end{aligned}$$

Therefore,

$$\mathbb{E}(Z_{n+1}|\mathcal{F}_n) = \mathbb{E}(Z_n \cdot (\operatorname{sech}\theta) e^{\theta Y_{n+1}}|\mathcal{F}_n) = Z_n \mathbb{E}((\operatorname{sech}\theta) e^{\theta Y_{n+1}}|\mathcal{F}_n) = Z_n \mathbb{E}((\operatorname{sech}\theta) e^{\theta Y_{n+1}})$$

where the second equality follows by "taking out what is known" and the third equality follows by independence. The final step is to compute $\mathbb{E}((\operatorname{sech}\theta) e^{\theta Y_{n+1}})$. Note that

$$\mathbb{E}(e^{\theta Y_{n+1}}) = e^{\theta \cdot 1} \cdot \frac{1}{2} + e^{\theta \cdot -1} \cdot \frac{1}{2} = \frac{e^\theta + e^{-\theta}}{2} = \frac{1}{\operatorname{sech}\theta}$$

and so

$$\mathbb{E}((\operatorname{sech}\theta) e^{\theta Y_{n+1}}) = (\operatorname{sech}\theta) \mathbb{E}(e^{\theta Y_{n+1}}) = (\operatorname{sech}\theta) \cdot \frac{1}{\operatorname{sech}\theta} = 1.$$

In other words, we have shown that

$$\mathbb{E}(Z_{n+1}|\mathcal{F}_n) = Z_n$$

which implies that $\{Z_n, n = 0, 1, 2 \ldots\}$ is a martingale.

The following two examples give more martingales derived from simple random walk.

Example 4.8. As in the previous example, let $Y_1, Y_2, \ldots$ be independent and identically distributed random variables with $\mathbf{P}\{Y_1 = 1\} = \mathbf{P}\{Y_1 = -1\} = \frac{1}{2}$, set $S_0 = 0$, and for $n = 1, 2, 3, \ldots$, define the random variable S_n by $S_n = Y_1 + \cdots + Y_n$ so that $\{S_n, n = 0, 1, 2, \ldots\}$ is a simple random walk starting at 0. Define the process $\{M_n, n = 0, 1, 2, \ldots\}$ by setting

$$M_n = S_n^3 - 3nS_n.$$

Show that $\{M_n, n = 0, 1, 2, \ldots\}$ is a martingale.

Solution. If $M_n = S_n^3 - 3nS_n$, then

$$\begin{aligned} M_{n+1} &= S_{n+1}^3 - 3(n+1)S_{n+1} \\ &= (S_n + Y_{n+1})^3 - 3(n+1)(S_n + Y_{n+1}) \\ &= S_n^3 + 3S_n^2 Y_{n+1} + 3S_n Y_{n+1}^2 + Y_{n+1}^3 - 3(n+1)S_n - 3(n+1)Y_{n+1} \\ &= M_n + 3S_n(Y_{n+1}^2 - 1) + 3S_n^2 Y_{n+1} - 3(n+1)Y_{n+1} + Y_{n+1}^3. \end{aligned}$$

Thus, we see that we will be able to conclude that $\{M_n, n = 0, 1, \ldots\}$ is a martingale if we can show that

$$\mathbb{E}\left(3S_n(Y_{n+1}^2 - 1) + 3S_n^2 Y_{n+1} - 3(n+1)Y_{n+1} + Y_{n+1}^3 | \mathcal{F}_n\right) = 0.$$

Now

$$3\mathbb{E}(S_n(Y_{n+1}^2 - 1)|\mathcal{F}_n) = 3S_n\mathbb{E}(Y_{n+1}^2 - 1) \quad \text{and} \quad 3\mathbb{E}(S_n^2 Y_{n+1}|\mathcal{F}_n) = 3S_n^2\mathbb{E}(Y_{n+1})$$

by "taking out what is known," and using the fact that Y_{n+1} and $\mathcal{F}_n$ are independent. Furthermore,

$$3(n+1)\mathbb{E}(Y_{n+1}|\mathcal{F}_n) = 3(n+1)\mathbb{E}(Y_{n+1}) \quad \text{and} \quad \mathbb{E}(Y_{n+1}^3|\mathcal{F}_n) = \mathbb{E}(Y_{n+1}^3)$$

using the fact that Y_{n+1} and $\mathcal{F}_n$ are independent. Since $\mathbb{E}(Y_{n+1}) = 0$, $\mathbb{E}(Y_{n+1}^2) = 1$, and $\mathbb{E}(Y_{n+1}^3) = 0$, we see that

$$\begin{aligned} \mathbb{E}(M_{n+1}|\mathcal{F}_n) &= M_n + 3S_n\mathbb{E}(Y_{n+1}^2 - 1) + 3S_n^2\mathbb{E}(Y_{n+1}) - 3(n+1)\mathbb{E}(Y_{n+1}) + \mathbb{E}(Y_{n+1}^3) \\ &= M_n + 3S_n \cdot (1-1) + 3S_n^2 \cdot 0 - 3(n+1) \cdot 0 + 0 \\ &= M_n \end{aligned}$$

which proves that $\{M_n, n = 0, 1, 2, \ldots\}$ is, in fact, a martingale.

The following example is the most important discrete-time martingale calculation that you will do. The process $\{I_j, j = 0, 1, 2, \ldots\}$ defined below is an example of a discrete stochastic integral. In fact, stochastic integration is one of the greatest achievements of 20th century probability and, as we will see, is fundamental to the mathematical theory of finance and option pricing.

Example 4.9. As in the previous example, let $Y_1, Y_2, \ldots$ be independent and identically distributed random variables with $\mathbf{P}\{Y_1 = 1\} = \mathbf{P}\{Y_1 = -1\} = \frac{1}{2}$, set $S_0 = 0$, and for $n = 1, 2, 3, \ldots$, define the random variable S_n by $S_n = Y_1 + \cdots + Y_n$ so that $\{S_n, n = 0, 1, 2, \ldots\}$ is a simple random walk starting at 0. Now suppose that $I_0 = 0$ and for $j = 1, 2, \ldots$ define I_j to be

$$I_j = \sum_{n=1}^{j} S_{n-1}(S_n - S_{n-1}).$$

Prove that $\{I_j, j = 0, 1, 2, \ldots\}$ is a martingale.

Solution. If

$$I_j = \sum_{n=1}^{j} S_{n-1}(S_n - S_{n-1}).$$

then

$$I_{j+1} = I_j + S_j(S_{j+1} - S_j).$$

Therefore,

$$\begin{aligned}\mathbb{E}(I_{j+1}|\mathcal{F}_j) = \mathbb{E}(I_j + S_j(S_{j+1} - S_j)|\mathcal{F}_j) &= \mathbb{E}(I_j|\mathcal{F}_j) + \mathbb{E}(S_j(S_{j+1} - S_j)|\mathcal{F}_j) \\ &= I_j + S_j\mathbb{E}(S_{j+1}|\mathcal{F}_j) - S_j^2\end{aligned}$$

where we have "taken out what is known" three times. Furthermore, since $\{S_j, j = 0, 1, \ldots\}$ is a martingale,

$$\mathbb{E}(S_{j+1}|\mathcal{F}_j) = S_j.$$

Combining everything gives

$$\mathbb{E}(I_{j+1}|\mathcal{F}_j) = I_j + S_j\mathbb{E}(S_{j+1}|\mathcal{F}_j) - S_j^2 = I_j + S_j^2 - S_j^2 = I_j$$

which proves that $\{I_j, j = 0, 1, 2, \ldots\}$ is, in fact, a martingale.

Exercise 4.10. Suppose that $\{I_j, j = 0, 1, 2, \ldots\}$ is defined as in the previous example. Show that

$$\mathrm{Var}(I_j) = \frac{j(j-1)}{2}$$

for all $j = 0, 1, 2, \ldots$.

This next example gives several martingales derived from *biased random walk*.

Example 4.11. Suppose that $Y_1, Y_2, \ldots$ are independent and identically distributed random variables with $\mathbf{P}\{Y_1 = 1\} = p$, $\mathbf{P}\{Y_1 = -1\} = 1 - p$ for some $0 < p < 1/2$. Let $S_n = Y_1 + \cdots + Y_n$ denote their partial sums so that $\{S_n, n = 0, 1, 2, \ldots\}$ is a *biased random walk*. (Note that $\{S_n, n = 0, 1, 2, \ldots\}$ is no longer a *simple* random walk.)

(a) Show that $X_n = S_n - n(2p - 1)$ is a martingale.

(b) Show that $M_n = X_n^2 - 4np(1-p) = [S_n - n(2p-1)]^2 - 4np(1-p)$ is a martingale.

(c) Show that $Z_n = \left(\frac{1-p}{p}\right)^{S_n}$ is a martingale.

Solution. We begin by noting that

$$\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n) = \sigma(S_0, \ldots, S_n) = \sigma(X_0, \ldots, X_n) = \sigma(M_0, \ldots, M_n) = \sigma(Z_0, \ldots, Z_n).$$

(a) The first step is to calculate $\mathbb{E}(Y_1)$. That is,

$$\mathbb{E}(Y_1) = 1 \cdot \mathbf{P}\{Y = 1\} + (-1) \cdot \mathbf{P}\{Y = -1\} = p - (1 - p) = 2p - 1.$$

Since $S_{n+1} = S_n + Y_{n+1}$, we see that

$$\begin{aligned}\mathbb{E}(S_{n+1}|\mathcal{F}_n) = \mathbb{E}(S_n + Y_{n+1}|\mathcal{F}_n) &= \mathbb{E}(S_n|\mathcal{F}_n) + \mathbb{E}(Y_{n+1}|\mathcal{F}_n)\\ &= S_n + \mathbb{E}(Y_{n+1})\\ &= S_n + 2p - 1\end{aligned}$$

by "taking out what is known" and using the fact that Y_{n+1} and $\mathcal{F}_n$ are independent. This implies that

$$\begin{aligned}\mathbb{E}(X_{n+1}|\mathcal{F}_n) = \mathbb{E}(S_{n+1} - (n+1)(2p-1)|\mathcal{F}_n) &= \mathbb{E}(S_{n+1}|\mathcal{F}_n) - (n+1)(2p-1)\\ &= S_n + 2p - 1 - (n+1)(2p-1)\\ &= S_n - n(2p-1)\\ &= X_n,\end{aligned}$$

and so we conclude that $\{X_n, n = 1, 2, \ldots\}$ is, in fact, a martingale.

(b) Notice that we can write X_{n+1} as

$$\begin{aligned}X_{n+1} = S_{n+1} - (n+1)(2p-1) &= S_n + Y_{n+1} - n(2p-1) - (2p-1)\\ &= X_n + Y_{n+1} - (2p-1)\end{aligned}$$

and so

$$\begin{aligned}X_{n+1}^2 &= (X_n + Y_{n+1})^2 + (2p-1)^2 - 2(2p-1)(X_n + Y_{n+1})\\ &= X_n^2 + Y_{n+1}^2 + 2X_nY_{n+1} + (2p-1)^2 - 2(2p-1)(X_n + Y_{n+1}).\end{aligned}$$

Thus,

$$\begin{aligned}&\mathbb{E}(X_{n+1}^2|\mathcal{F}_n)\\ &\quad= \mathbb{E}(X_n^2|\mathcal{F}_n) + \mathbb{E}(Y_{n+1}^2|\mathcal{F}_n) + 2\mathbb{E}(X_nY_{n+1}|\mathcal{F}_n) + (2p-1)^2 - 2(2p-1)\mathbb{E}(X_n + Y_{n+1}|\mathcal{F}_n)\\ &\quad= X_n^2 + \mathbb{E}(Y_{n+1})^2 + 2X_n\mathbb{E}(Y_{n+1}) + (2p-1)^2 - 2(2p-1)(X_n + \mathbb{E}(Y_{n+1}))\\ &\quad= X_n^2 + 1 + 2(2p-1)X_n + (2p-1)^2 - 2(2p-1)(X_n + (2p-1))\\ &\quad= X_n^2 + 1 + 2(2p-1)X_n + (2p-1)^2 - 2(2p-1)X_n - 2(2p-1)^2\\ &\quad= X_n^2 + 1 - (2p-1)^2,\end{aligned}$$

by again "taking out what is known" and using the fact that Y_{n+1} and $\mathcal{F}_n$ are independent. Hence, we find

$$\begin{aligned}\mathbb{E}(M_{n+1}|\mathcal{F}_n) &= \mathbb{E}(X_{n+1}^2|\mathcal{F}_n) - 4(n+1)p(1-p)\\ &= X_n^2 + 1 - (2p-1)^2 - 4(n+1)p(1-p)\\ &= X_n^2 + 1 - (4p^2 - 4p + 1) - 4np(1-p) - 4p(1-p)\\ &= X_n^2 + 1 - 4p^2 + 4p - 1 - 4np(1-p) - 4p + 4p^2\\ &= X_n^2 - 4np(1-p)\\ &= M_n\end{aligned}$$

so that $\{M_n, n = 1, 2, \dots\}$ is, in fact, a martingale.

(c) Notice that

$$Z_{n+1} = \left(\frac{1-p}{p}\right)^{S_{n+1}} = \left(\frac{1-p}{p}\right)^{S_n+Y_{n+1}} = \left(\frac{1-p}{p}\right)^{S_n}\left(\frac{1-p}{p}\right)^{Y_{n+1}} = Z_n\left(\frac{1-p}{p}\right)^{Y_{n+1}}.$$

Therefore,

$$\begin{aligned}\mathbb{E}(Z_{n+1}|\mathcal{F}_n) = \mathbb{E}\left(Z_n\left(\frac{1-p}{p}\right)^{Y_{n+1}}\middle|\mathcal{F}_n\right) &= Z_n\mathbb{E}\left(\left(\frac{1-p}{p}\right)^{Y_{n+1}}\middle|\mathcal{F}_n\right)\\ &= Z_n\mathbb{E}\left(\left(\frac{1-p}{p}\right)^{Y_{n+1}}\right)\end{aligned}$$

where the second equality follows from "taking out what is known" and the third equality follows from the fact that Y_{n+1} and $\mathcal{F}_n$ are independent. We now compute

$$\mathbb{E}\left(\left(\frac{1-p}{p}\right)^{Y_{n+1}}\right) = p\left(\frac{1-p}{p}\right)^{1} + (1-p)\left(\frac{1-p}{p}\right)^{-1} = (1-p) + p = 1$$

and so we conclude

$$\mathbb{E}(Z_{n+1}|\mathcal{F}_n) = Z_n.$$

Hence, $\{Z_n, n = 0, 1, 2, \dots\}$ is, in fact, a martingale.

We now conclude this section with one final example. Although it is unrelated to simple random walk, it is an easy martingale calculation and is therefore worth including. In fact, it could be considered as a generalization of (c) of the previous example.

Example 4.12. Suppose that $Y_1, Y_2, \dots$ are independent and identically distributed random variables with $\mathbb{E}(Y_1) = 1$. Suppose further that $X_0 = Y_0 = 1$ and for $n = 1, 2, \dots$, let

$$X_n = Y_1 \cdot Y_2 \cdots Y_n = \prod_{j=1}^{n} Y_j.$$

Verify that $\{X_n, n = 0, 1, 2, \dots\}$ is a martingale with respect to $\{\mathcal{F}_n = \sigma(Y_0, \dots, Y_n), n = 0, 1, 2, \dots\}$.

Solution. We find

$$\begin{aligned}\mathbb{E}(X_{n+1}|\mathcal{F}_n) &= \mathbb{E}(X_n \cdot Y_{n+1}|\mathcal{F}_n)\\ &= X_n\mathbb{E}(Y_{n+1}|\mathcal{F}_n) \quad \text{(by taking out what is known)}\\ &= X_n\mathbb{E}(Y_{n+1}) \quad \text{(since } Y_{n+1} \text{ is independent of } \mathcal{F}_n\text{)}\\ &= X_n \cdot 1\\ &= X_n\end{aligned}$$

and so $\{X_n, n = 0, 1, 2, \dots\}$ is, in fact, a martingale.