Statistics 441 (Fall 2014) October 29, 31, November 3, 2014
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Lecture #23, 24, 25: Deriving the Black–Scholes Partial Differential Equation

Our goal for today is to use Itô's formula to derive the Black-Scholes partial differential equation. We will then solve this equation next lecture.

Recall from Lecture #2 that $D(t)$ denotes the value at time t of an investment which grows according to a continuously compounded interest rate r. We know its value at time $t \geq 0$ is given by $D(t) = e^{rt} D_0$ which is the solution to the differential equation $D'(t) = rD(t)$ with initial condition $D(0) = D_0$. Written in differential form, this becomes

$$\mathrm{d}D(t) = rD(t)\,\mathrm{d}t. \tag{15.1}$$

We now assume that our stock price is modelled by geometric Brownian motion. That is, let S_t denote the price of the stock at time t, and assume that S_t satisfies the stochastic differential equation

$$\mathrm{d}S_t = \sigma S_t\,\mathrm{d}B_t + \mu S_t\,\mathrm{d}t. \tag{15.2}$$

We can check using Version II of Itô's formula (Theorem 14.9) that the solution to this SDE is geometric Brownian motion $\{S_t, t \geq 0\}$ given by

$$S_t = S_0 \exp\left\{\sigma B_t + \left(\mu - \frac{\sigma^2}{2}\right) t\right\}$$

where S_0 is the initial value.

Remark. There are two, equally common, ways to parametrize the drift of the geometric Brownian motion. The first is so that the process is simpler,

$$S_t = S_0 \exp\left\{\sigma B_t + \mu t\right\},$$

and leads to the more complicated SDE

$$\mathrm{d}S_t = \sigma S_t\,\mathrm{d}B_t + \left(\mu + \frac{\sigma^2}{2}\right) S_t\,\mathrm{d}t.$$

The second is so that the SDE is simpler,

$$\mathrm{d}S_t = \sigma S_t\,\mathrm{d}B_t + \mu S_t\,\mathrm{d}t,$$

and leads to the more complicated process

$$S_t = S_0 \exp\left\{\sigma B_t + \left(\mu - \frac{\sigma^2}{2}\right) t\right\}.$$

We choose the parametrization given by (15.2) to be consistent with Higham [11].

We also recall from Lecture #1 the definition of a European call option.

Definition 15.1. A *European call option* with strike price E at time T gives its holder an opportunity (i.e., the right, but not the obligation) to buy from the writer one share of the prescribed stock at time T for price E.

Notice that if, at time T, the value of the stock is less than E, then the option is worthless and will not be exercised, but if the value of the stock is greater than E, then the option is valuable and will therefore be exercised.

That is,

- if $S_T \le E$, then the option is worthless, but
- if $S_T > E$, then the option has the value $S_T - E$.

Thus, the value of the option at time T is $(S_T - E)^+ = \max\{0, S_T - E\}$. Our goal, therefore, is to determine the value of this option at time 0.

We will write V to denote the value of the option. Since V depends on both time and on the underlying stock, we see that $V(t, S_t)$ denotes the value of the option at time t, $0 \le t \le T$.

Hence,

- $V(T, S_T) = (S_T - E)^+$ is the value of the option at the expiry time T, and
- $V(0, S_0)$ denotes the value of option at time 0.

Example 15.2. Assuming that the function $V \in C^1([0,\infty)) \times C^2(\mathbb{R})$, use Itô's formula on $V(t, S_t)$ to compute $\mathrm{d}V(t, S_t)$.

Solution. By Version IV of Itô's formula (Theorem 14.12), we find

$$\mathrm{d}V(t, S_t) = \dot{V}(t, S_t)\,\mathrm{d}t + V'(t, S_t)\,\mathrm{d}S_t + \frac{1}{2}V''(t, S_t)\,\mathrm{d}\langle S\rangle_t.$$

From (15.2), the SDE for geometric Brownian motion is

$$\mathrm{d}S_t = \sigma S_t\,\mathrm{d}B_t + \mu S_t\,\mathrm{d}t$$

and so we find

$$\mathrm{d}\langle S\rangle_t = (\mathrm{d}S_t)^2 = \sigma^2 S_t^2\,\mathrm{d}t$$

using the rules $(\mathrm{d}B_t)^2 = \mathrm{d}t$, $(\mathrm{d}t)^2 = (\mathrm{d}B_t)(\mathrm{d}t) = (\mathrm{d}t)(\mathrm{d}B_t) = 0$. Hence, we conclude

$$\begin{aligned}\mathrm{d}V(t, S_t) &= \dot{V}(t, S_t)\,\mathrm{d}t + V'(t, S_t)\big[\sigma S_t\,\mathrm{d}B_t + \mu S_t\,\mathrm{d}t\big] + \frac{1}{2}V''(t, S_t)\big[\sigma^2 S_t^2\,\mathrm{d}t\big]\\ &= \sigma S_t V'(t, S_t)\,\mathrm{d}B_t + \left[\dot{V}(t, S_t) + \mu S_t V'(t, S_t) + \frac{\sigma^2}{2} S_t^2 V''(t, S_t)\right]\mathrm{d}t. \qquad (15.3)\end{aligned}$$

We now recall the no arbitrage assumption from Lecture #2 which states that "there is never an opportunity to make a risk-free profit that gives a greater return than that provided by interest from a bank deposit."

Thus, to find the fair value of the option $V(t, S_t)$, $0 \le t \le T$, we will set up a *replicating portfolio* of assets and bonds that has precisely the *same risk* at time t as the option does at time t. The portfolio consists of a cash deposit D and a number A of assets.

We assume that we can vary the number of assets and the size of our cash deposit at time t so that both D and A are allowed to be functions of both the time t and the asset price S_t. (Technically, our *trading strategy* needs to be previsible; we can only alter our portfolio depending on what has happened already.)

That is, if Π denotes our portfolio, then the value of our portfolio at time t is given by

$$\Pi(t, S_t) = A(t, S_t)S_t + D(t, S_t). \tag{15.4}$$

Recall that we are allowed to short-sell both the stocks and the bonds and that there are no transaction costs involved. Furthermore, it is worth noting that, although our strategy for buying bonds may depend on both the time and the behaviour of the stock, the bond is *still* a risk-free investment which evolves according to (15.1) as

$$\mathrm{d}D(t, S_t) = rD(t, S_t)\,\mathrm{d}t. \tag{15.5}$$

The assumption that the portfolio is replicating means precisely that the portfolio is *self-financing*; in other words, the value of the portfolio one time step later is financed entirely by the current wealth. In terms of stochastic differentials, the self-financing condition is

$$\mathrm{d}\Pi(t, S_t) = A(t, S_t)\,\mathrm{d}S_t + \,\mathrm{d}D(t, S_t),$$

which, using (15.2) and (15.5), is equivalent to

$$\begin{aligned}\mathrm{d}\Pi(t, S_t) &= A(t, S_t)\big[\sigma S_t\,\mathrm{d}B_t + \mu S_t\,\mathrm{d}t\big] + rD(t, S_t)\,\mathrm{d}t \\ &= \sigma A(t, S_t)S_t\,\mathrm{d}B_t + \big[\mu A(t, S_t)S_t + rD(t, S_t)\big]\,\mathrm{d}t.\end{aligned} \tag{15.6}$$

The final step is to consider $V(t, S_t) - \Pi(t, S_t)$. By the no arbitrage assumption, the change in $V(t, S_t) - \Pi(t, S_t)$ over any time step is non-random. Furthermore, it must equal the corresponding growth offered by the continuously compounded risk-free interest rate. In terms of differentials, if we write

$$U_t = V(t, S_t) - \Pi(t, S_t),$$

then U_t must be non-random and grow according to (15.1) so that

$$\mathrm{d}U_t = rU_t\,\mathrm{d}t.$$

That is,

$$\mathrm{d}\big[V(t, S_t) - \Pi(t, S_t)\big] = r\big[V(t, S_t) - \Pi(t, S_t)\big]\,\mathrm{d}t. \tag{15.7}$$

The logic is outlined by Higham [11] on page 79.

Using (15.3) for $\mathrm{d}V(t,S_t)$ and (15.6) for $\mathrm{d}\Pi(t,S_t)$, we find

$$\begin{aligned}
&\mathrm{d}\big[V(t,S_t)-\Pi(t,S_t)\big]\\
&\quad=\left(\sigma S_t V'(t,S_t)\,\mathrm{d}B_t+\left[\dot V(t,S_t)+\mu S_t V'(t,S_t)+\frac{\sigma^2}{2}S_t^2V''(t,S_t)\right]\mathrm{d}t\right)\\
&\qquad\qquad-\left(\sigma A(t,S_t)S_t\,\mathrm{d}B_t+\big[\mu A(t,S_t)S_t+rD(t,S_t)\big]\,\mathrm{d}t\right)\\
&\quad=\sigma S_t\big[V'(t,S_t)-A(t,S_t)\big]\,\mathrm{d}B_t\\
&\qquad\qquad+\left[\dot V(t,S_t)+\mu S_t V'(t,S_t)+\frac{\sigma^2}{2}S_t^2V''(t,S_t)-\mu A(t,S_t)S_t-rD(t,S_t)\right]\mathrm{d}t\\
&\quad=\sigma S_t\big[V'(t,S_t)-A(t,S_t)\big]\,\mathrm{d}B_t\\
&\qquad\qquad+\left[\dot V(t,S_t)+\frac{\sigma^2}{2}S_t^2V''(t,S_t)-rD(t,S_t)+\mu S_t\big[V'(t,S_t)-A(t,S_t)\big]\right]\mathrm{d}t. \qquad (15.8)
\end{aligned}$$

Since we assume that the change over any time step is non-random, it must be the case that the $\mathrm{d}B_t$ term is 0. In order for the $\mathrm{d}B_t$ term to be 0, we simply choose

$$A(t,S_t)=V'(t,S_t).$$

This means that that $\mathrm{d}t$ term

$$\dot V(t,S_t)+\frac{\sigma^2}{2}S_t^2V''(t,S_t)-rD(t,S_t)+\mu S_t\big[V'(t,S_t)-A(t,S_t)\big]$$

reduces to

$$\dot V(t,S_t)+\frac{\sigma^2}{2}S_t^2V''(t,S_t)-rD(t,S_t)$$

since we already need $A(t,S_t)=V'(t,S_t)$ for the $\mathrm{d}B_t$ piece. Looking at (15.7) therefore gives

$$\dot V(t,S_t)+\frac{\sigma^2}{2}S_t^2V''(t,S_t)-rD(t,S_t)=r\big[V(t,S_t)-\Pi(t,S_t)\big]. \qquad (15.9)$$

Using the facts that

$$\Pi(t,S_t)=A(t,S_t)S_t+D(t,S_t)$$

and

$$A(t,S_t)=V'(t,S_t)$$

therefore imply that (15.9) becomes

$$\dot V(t,S_t)+\frac{\sigma^2}{2}S_t^2V''(t,S_t)-rD(t,S_t)=rV(t,S_t)-rS_tV'(t,S_t)-rD(t,S_t)$$

which, upon simplification, reduces to

$$\dot V(t,S_t)+\frac{\sigma^2}{2}S_t^2V''(t,S_t)+rS_tV'(t,S_t)-rV(t,S_t)=0.$$

In other words, we must find a function $V(t,x)$ which satisfies the *Black-Scholes partial differential equation*

$$\dot{V}(t,x) + \frac{\sigma^2}{2}x^2V''(t,x) + rxV'(t,x) - rV(t,x) = 0. \tag{15.10}$$

Remark. We have finally arrived at what Higham [11] calls "the famous *Black-Scholes* partial differential equation (PDE)" given by equation (8.15) on page 79.

We now mention two important points.

- The drift parameter μ in the asset model does NOT appear in the Black-Scholes PDE.
- Actually, we have not yet specified what type of option is being valued. The PDE given in (15.10) must be satisfied by ANY option on the asset S whose value can be expressed as a smooth function, i.e., a function in $C^1([0,\infty)) \times C^2(\mathbb{R})$.

In view of the second item, we really want to price a European call option with strike price E. This amounts to requiring $V(T, S_T) = (S_T - E)^+$. Our goal, therefore, in the next lecture is to solve the Black-Scholes partial differential equation

$$\dot{V}(t,x) + \frac{\sigma^2}{2}x^2V''(t,x) + rxV'(t,x) - rV(t,x) = 0$$

for $V(t,x)$, $0 \le t \le T$, $x \in \mathbb{R}$, *subject to the boundary condition*

$$V(T,x) = (x-E)^+.$$