Statistics 441 (Fall 2014) October 10, 15, 2014
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Lecture #16, 17: Itô Integration (Part II)

Recall from last lecture that we defined the Itô integral of Brownian motion as

$$\begin{aligned}\int_0^t B_s \, \mathrm{d}B_s &= \lim L_n \text{ in } L^2 \\ &= \frac{1}{2}B_t^2 - \frac{t}{2}. \end{aligned} \tag{12.1}$$

where $\{\pi_n, n = 1, 2, \ldots\}$ is a refinement of $[0, t]$ with $\operatorname{mesh}(\pi_n) \to 0$ and

$$L_n = \sum_{i=1}^{n} B_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})$$

denotes the left-hand Riemann sum corresponding to the partition $\pi_n = \{0 = t_0 < t_1 < \cdots < t_n = t\}$.

We saw that the definition of I_t depended on the intermediate point used in the Riemann sum, and that the reason for *choosing* the left-hand sum was that it produced a martingale.

We now present another example which shows some of the dangers of a naïve attempt at stochastic integration.

Example 12.1. Let $\{B_t, \ t \ge 0\}$ be a realization of Brownian motion with $B_0 = 0$, and suppose that for any fixed $0 \le t < 1$ we define the random variable I_t by

$$I_t = \int_0^t B_1 \, \mathrm{d}B_s.$$

Since B_1 is constant (for a given realization), we might expect that

$$I_t = \int_0^t B_1 \, \mathrm{d}B_s = B_1 \int_0^t \mathrm{d}B_s = B_1(B_t - B_0) = B_1 B_t.$$

However,

$$\mathbb{E}(I_t) = \mathbb{E}(B_1 B_t) = \min\{1, t\} = t$$

which is not constant. Therefore, if we want to obtain martingales, this is not how we should define the integral I_t. The problem here is that the random variable B_1 is not adapted to $\mathcal{F}_t = \sigma(B_s, \ 0 \le s \le t)$ for any fixed $0 \le t < 1$.

From the previous example, we see that in order to define

$$I_t = \int_0^t g(s) \, \mathrm{d}B_s$$

the stochastic process $\{g(s), \ 0 \le s \le t\}$ will necessarily need to be adapted to the Brownian filtration $\{\mathcal{F}_s, \ 0 \le s \le t\} = \{\sigma(B_r, \ 0 \le r \le s), \ 0 \le s \le t\}$.

Definition 12.2. Let L^2_{ad} denote the space of stochastic processes $g = \{g(t), t \geq 0\}$ such that

- g is adapted to the Brownian filtration $\{\mathcal{F}_t,\ t \geq 0\}$ (i.e., $g(t) \in \mathcal{F}_t$ for every $t > 0$), and
- $\displaystyle\int_0^T \mathbb{E}[g^2(t)]\,\mathrm{d}t < \infty$ for every $T > 0$.

Our goal is to now define

$$I_t(g) = \int_0^t g(s)\,\mathrm{d}B_s$$

for $g \in L^2_{\text{ad}}$. This is accomplished in a more technical manner than the construction of the Wiener integral, and the precise details will therefore be omitted. Complete details may by found in [12], however.

The first step involves defining the integral for *step stochastic processes*, and the second step is to then pass to a limit.

Suppose that $g = \{g(t), t \geq 0\}$ is a stochastic process. We say that g is a *step stochastic process* if for every $t \geq 0$ we can write

$$g(s,\omega) = \sum_{i=1}^{n-1} X_{i-1}(\omega)\mathbb{1}_{[t_{i-1},t_i)}(s) + X_{n-1}(\omega)\mathbb{1}_{[t_{n-1},t_n]}(s) \tag{12.2}$$

for $0 \leq s \leq t$ where $\{0 = t_0 < t_1 < \cdots < t_n = t\}$ is a partition of $[0,t]$ and $\{X_j, j = 0, 1, \ldots, n-1\}$ is a finite collection of random variables. Define the integral of such a g as

$$I_t(g)(\omega) = \int_0^t g(s,\omega)\,\mathrm{d}B_s(\omega) = \sum_{i=1}^{n} X_{i-1}(\omega)(B_{t_i}(\omega) - B_{t_{i-1}}(\omega)), \tag{12.3}$$

and note that (12.3) is simply a discrete stochastic integral as in (5.1), the so-called *martingale transform* of X by B.

The second, and more difficult, step is show that it is possible to approximate an arbitrary $g \in L^2_{\text{ad}}$ by a sequence of step processes $g_n \in L^2_{\text{ad}}$ such that

$$\lim_{n\to\infty} \int_0^t \mathbb{E}(|g_n(s) - g(s)|^2)\,\mathrm{d}s = 0.$$

We then define $I_t(g)$ to be the limit in L^2 of the approximating Itô integrals $I_t(g_n)$ defined by (12.3), and show that the limit does not depend of the choice of step processes $\{g_n\}$; that is,

$$I_t(g) = \lim_{n\to\infty} I_t(g_n) \quad \text{in } L^2 \tag{12.4}$$

and so we have the following definition.

Definition 12.3. If $g \in L^2_{\text{ad}}$, define the Itô integral of g to be

$$I_t(g) = \int_0^t g(s)\,\mathrm{d}B_s$$

where $I_t(g)$ is defined as the limit in (12.4).

Notice that the definition of the Itô integral did not use any approximating Riemann sums. However, last lecture we calculated $\int_0^t B_s\,\mathrm{d}B_s$ directly by taking the limit in L^2 of the approximating Riemann sums. It is important to know when both approaches give the same answer which is the content of the following theorem. For a proof, see Theorem 4.7.1 of [12].

Theorem 12.4. *If the stochastic process $g \in L^2_{ad}$ and $\mathbb{E}(g(s)g(t))$ is a continuous function of s and t, then*

$$\int_0^t g(s)\,\mathrm{d}B_s = \lim \sum_{i=1}^n g(t_{i-1})(B_{t_i} - B_{t_{i-1}}) \quad \textit{in } L^2.$$

Example 12.5. For example, if the stochastic process g is a Brownian motion, then B_t is necessarily $\mathcal{F}_t$-measurable with $\mathbb{E}(B_t^2) = t < \infty$ for every $t > 0$. Since $\mathbb{E}(B_sB_t) = \min\{s,t\}$ is a continuous function of s and t, we conclude that Theorem 12.4 can be applied to calculate $\int_0^t B_s\,\mathrm{d}B_s$. This is exactly what we did in (11.6).

The following result collects together a number of properties of the Itô integral. It is relatively straightforward to prove all of these properties when g is a step stochastic process. It is rather more involved to pass to the appropriate limits to obtain these results for general $g \in L^2_{\text{ad}}$.

Theorem 12.6. *Suppose that g, $h \in L^2_{ad}$, and let*

$$I_t(g) = \int_0^t g(s)\,\mathrm{d}B_s \quad \textit{and} \quad I_t(h) = \int_0^t h(s)\,\mathrm{d}B_s.$$

- *If α, $\beta \in \mathbb{R}$ are constants, then $I_t(\alpha g + \beta h) = \alpha I_t(g) + \beta I_t(h)$.*
- *$I_t(g)$ is a random variable with $I_0(g) = 0$, $\mathbb{E}(I_t(g)) = 0$ and*

$$\mathrm{Var}(I_t(g)) = \mathbb{E}[I_t^2(g)] = \int_0^t \mathbb{E}[g^2(s)]\,\mathrm{d}s. \tag{12.5}$$

- *The covariance of $I_t(g)$ and $I_t(h)$ is given by*

$$\mathbb{E}[I_t(g)I_t(h)] = \int_0^t \mathbb{E}[g(s)h(s)]\,\mathrm{d}s.$$

- *The process $\{I_t,\ t \geq 0\}$ is a martingale with respect to the Brownian filtration.*
- *The trajectory $t \mapsto I_t$ is a continuous function of t.*

Remark. The equality (12.5) in the second part of this theorem is sometimes known as the *Itô isometry.*

Remark. It is important to observe that the Wiener integral is a special case of the Itô integral. That is, if g is a bounded, piecewise continuous deterministic $L^2([0,\infty))$ function, then $g \in L^2_{\text{ad}}$ and so the Itô integral of g with respect to Brownian motion can be constructed. The fact that g is deterministic means that we recover the properties for the Wiener integral from the properties in Theorem 12.6 for the Itô integral. Theorem 9.2, the integration-by-parts formula for Wiener integration, will follow from the generalized version of Itô's formula (which we will start to discuss next lecture).

Remark. It is also important to observe that, unlike the Wiener integral, there is no general form of the distribution of $I_t(g)$. In general, the Riemann sum approximations to $I_t(g)$ contain terms of the form

$$g(t_{i-1})(B_{t_i} - B_{t_{i-1}}). \tag{12.6}$$

When g is deterministic, the distribution of the $I_t(g)$ is normal as a consequence of the fact that the sum of independent normals is normal. However, when g is random, the distribution of (12.6) is not necessarily normal. The following exercises illustrates this point.

Exercise 12.7. Consider

$$I = \int_0^1 B_s \, \mathrm{d}B_s = \frac{B_1^2}{2} - \frac{1}{2}.$$

Since $B_1 \sim \mathcal{N}(0,1)$, we know that $B_1^2 \sim \chi^2(1)$, and so we conclude that

$$2I + 1 \sim \chi^2(1).$$

Simulate 10000 realizations of I and plot a histogram of $2I+1$. Does your simulation match the theory?

Exercise 12.8. Suppose that $\{B_t, t \geq 0\}$ is a standard Brownian motion, and let the stochastic process $\{g(t), t \geq 0\}$ be defined as follows. At time $t = 0$, flip a fair coin and let $g(0) = 2$ if the coin shows heads, and let $g(0) = 3$ if the coin shows tails. At time $t = \sqrt{2}$, roll a fair die and let $g(\sqrt{2}\,)$ equal the number of dots showing on the die. If $0 < t < \sqrt{2}$, define $g(t) = g(0)$, and if $t > \sqrt{2}$, define $g(t) = g(\sqrt{2}\,)$. Note that $\{g(t), t \geq 0\}$ is a step stochastic process.

(a) Express g in the form (12.2).

(b) Sketch a graph of the stochastic process $\{g(t), t \geq 0\}$.

(c) Determine the mean and the variance of

$$\int_0^5 g(s) \, \mathrm{d}B_s.$$

(d) If possible, determine the distribution of

$$\int_0^5 g(s) \, \mathrm{d}B_s.$$