Lecture #14: Further Properties of the Wiener Integral

Recall that we have defined the Wiener integral of a bounded, piecewise continuous deterministic function in $L^2([0,\infty))$ with respect to Brownian motion as a normal random variable, namely

$$\int_0^t g(s)\,\mathrm{d}B_s \sim \mathcal{N}\left(0, \int_0^t g^2(s)\,\mathrm{d}s\right),$$

and that we have derived the integration-by-parts formula. That is, if $g : [0,\infty) \to \mathbb{R}$ is a bounded, continuous function in $L^2([0,\infty))$ such that g is differentiable with g' also bounded and continuous, then

$$\int_0^t g(s)\,\mathrm{d}B_s = g(t)B_t - \int_0^t g'(s)B_s\,\mathrm{d}s$$

holds as an equality in distribution of random variables. The purpose of today's lecture is to give some further properties of the Wiener integral.

Example 10.1. Recall from Example 9.4 that

$$\int_0^1 B_s\,\mathrm{d}s = B_1 - \int_0^1 s\,\mathrm{d}B_s.$$

We know from that example (or from Lecture #11) that

$$\int_0^1 B_s\,\mathrm{d}s \sim \mathcal{N}(0,1/3).$$

Furthermore, we know that $B_1 \sim \mathcal{N}(0,1)$, and we can easily calculate that

$$\int_0^1 s\,\mathrm{d}B_s \sim \mathcal{N}\left(0, \int_0^1 s^2\,\mathrm{d}s\right) = \mathcal{N}(0,1/3).$$

If B_1 and

$$\int_0^1 s\,\mathrm{d}B_s$$

were independent random variables, then from Exercise 3.12 the distribution of

$$B_1 - \int_0^1 s\,\mathrm{d}B_s$$

would be $\mathcal{N}(0,1+1/3) = \mathcal{N}(0,4/3)$. However,

$$B_1 - \int_0^1 s\,\mathrm{d}B_s = \int_0^1 B_s\,\mathrm{d}s$$

which we know is $\mathcal{N}(0,1/3)$. Thus, we are forced to conclude that B_1 and

$$\int_0^1 s\,\mathrm{d}B_s$$

are *not* independent.

Suppose that g and h are bounded, piecewise continuous functions in $L^2([0,\infty))$ and consider the random variables

$$I_t(g) = \int_0^t g(s)\,\mathrm{d}B_s$$

and

$$I_t(h) = \int_0^t h(s)\,\mathrm{d}B_s.$$

As the previous example suggests, these two random variables are not, in general, independent. Using linearity of the Wiener integral, we can now calculate their covariance. Since

$$I_t(g) = \int_0^t g(s)\,\mathrm{d}B_s \sim \mathcal{N}\left(0, \int_0^t g^2(s)\,\mathrm{d}s\right),$$

$$I_t(h) = \int_0^t h(s)\,\mathrm{d}B_s \sim \mathcal{N}\left(0, \int_0^t h^2(s)\,\mathrm{d}s\right),$$

and

$$I_t(g+h) = \int_0^t [g(s)+h(s)]\,\mathrm{d}B_s \sim \mathcal{N}\left(0, \int_0^t [g(s)+h(s)]^2\,\mathrm{d}s\right),$$

and since

$$\mathrm{Var}(I_t(g+h)) = \mathrm{Var}(I_t(g)+I_t(h)) = \mathrm{Var}(I_t(g)) + \mathrm{Var}(I_t(h)) + 2\,\mathrm{Cov}(I_t(g), I_t(h)),$$

we conclude that

$$\int_0^t [g(s)+h(s)]^2\,\mathrm{d}s = \int_0^t g^2(s)\,\mathrm{d}s + \int_0^t h^2(s)\,\mathrm{d}s + 2\,\mathrm{Cov}(I_t(g), I_t(h)).$$

Expanding the square on the left-side and simplifying implies that

$$\mathrm{Cov}(I_t(g), I_t(h)) = \int_0^t g(s)h(s)\,\mathrm{d}s.$$

Note that taking $g = h$ gives

$$\mathrm{Var}(I_t(g)) = \mathrm{Cov}(I_t(g), I_t(g)) = \int_0^t g(s)g(s)\,\mathrm{d}s = \int_0^t g^2(s)\,\mathrm{d}s$$

in agreement with our previous work. This suggests that the covariance formula should not come as a surprise to you!

Exercise 10.2. Suppose that $g(s) = \sin s$, $0 \le s \le \pi$, and $h(s) = \cos s$, $0 \le s \le \pi$.

(a) Show that $\mathrm{Cov}(I_\pi(g), I_\pi(h)) = 0$.

(b) Prove that $I_\pi(g)$ and $I_\pi(h)$ are independent. *Hint: Theorem 3.17 will be useful here.*

The same proof you used for (b) of the previous exercise holds more generally.

Theorem 10.3. *If g and h are bounded, piecewise continuous functions in $L^2([0,\infty))$ with*

$$\int_0^t g(s)h(s)\,\mathrm{d}s = 0,$$

then the random variables $I_t(g)$ and $I_t(h)$ are independent.

Exercise 10.4. Prove this theorem.

We end this lecture with two extremely important properties of the Wiener integral I_t, namely that $\{I_t, t \geq 0\}$ is a martingale and that the trajectories $t \mapsto I_t$ are continuous. The proof of the following theorem requires some facts about convergence in L^2 and is therefore beyond our present scope.

Theorem 10.5. *Suppose that $g : [0,\infty) \to \mathbb{R}$ is a bounded, piecewise continuous function in $L^2([0,\infty))$. If the process $\{I_t, t \geq 0\}$ is defined by setting $I_0 = 0$ and*

$$I_t = \int_0^t g(s)\,\mathrm{d}B_s$$

for $t > 0$, then

(a) *$\{I_t, t \geq 0\}$ is a continuous-time martingale with respect to the Brownian filtration $\{\mathcal{F}_t, t \geq 0\}$, and*

(b) *the trajectory $t \mapsto I_t$ is continuous.*

That is, $\{I_t, t \geq 0\}$ is a continuous-time continuous martingale.