Statistics 441 (Fall 2014) October 1, 2014
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Lecture #12: Wiener Integration

Having successfully determined the Riemann integral of Brownian motion, we will now learn how to integrate with respect to Brownian motion; that is, we will study the (yet-to-be-defined) stochastic integral

$$I_t = \int_0^t g(s)\,\mathrm{d}B_s.$$

Our experience with integrating Brownian motion suggests that I_t is really a random variable, and so one of our goals will be to determine the distribution of I_t.

Assume that g is bounded, piecewise continuous, and in $L^2([0,\infty))$, and suppose that we partition the interval $[0,t]$ by $0 = t_0 < t_1 < \cdots < t_n = t$. Consider the left-hand Riemann sum

$$\sum_{j=1}^n g(t_{j-1})(B_{t_j} - B_{t_{j-1}}).$$

Notice that our experience with the discrete stochastic integral suggests that we should choose a left-hand Riemann sum; that is, our discrete-time betting strategy Y_{j-1} needed to be previsible and so our continuous-time betting strategy $g(t)$ should also be previsible. When working with the Riemann sum, the previsible condition translates into taking the left-hand Riemann sum. We do, however, remark that when following a deterministic betting strategy, this previsible condition will turn out to not matter at all. On the other hand, when we follow a random betting strategy, it will be of the utmost importance.

To begin, let

$$I_t^{(n)} = \sum_{j=1}^n g(t_{j-1})(B_{t_j} - B_{t_{j-1}})$$

and notice that as in the discrete case, we can easily calculate $\mathbb{E}(I_t^{(n)})$ and $\mathrm{Var}(I_t^{(n)})$. Since $B_{t_j} - B_{t_{j-1}} \sim \mathcal{N}(0, t_j - t_{j-1})$, we have

$$\mathbb{E}(I_t^{(n)}) = \sum_{j=1}^n g(t_{j-1})\mathbb{E}(B_{t_j} - B_{t_{j-1}}) = 0,$$

and since the increments of Brownian motion are independent, we have

$$\mathrm{Var}(I_t^{(n)}) = \sum_{j=1}^n g^2(t_{j-1})\mathbb{E}(B_{t_j} - B_{t_{j-1}})^2 = \sum_{j=1}^n g^2(t_{j-1})(t_j - t_{j-1}).$$

We now make a crucial observation. The variance of $I_t^{(n)}$, namely

$$\sum_{j=1}^n g^2(t_{j-1})(t_j - t_{j-1}),$$

should look familiar. Since $0 = t_0 < t_1 < \cdots < t_n = t$ is a partition of $[0, t]$ we see that this sum is the left-hand Riemann sum approximating the Riemann integral

$$\int_0^t g^2(s)\,\mathrm{d}s.$$

We also see the reason to assume that g is bounded, piecewise continuous, and in $L^2([0,\infty))$. By Theorem 6.2, this condition is sufficient to guarantee that the limit

$$\lim_{n\to\infty} \sum_{j=1}^{n} g^2(t_{j-1})(t_j - t_{j-1})$$

exists and equals

$$\int_0^t g^2(s)\,\mathrm{d}s.$$

(Although by Theorem 6.3 it is possible to weaken the conditions on g, we will not concern ourselves with such matters.)

In summary, we conclude that

$$\lim_{n\to\infty} \mathbb{E}(I_t^{(n)}) = 0$$

and

$$\lim_{n\to\infty} \mathrm{Var}(I_t^{(n)}) = \int_0^t g^2(s)\,\mathrm{d}s.$$

Therefore, if we can somehow construct I_t as an appropriate limit of $I_t^{(n)}$, then it seems reasonable that $\mathbb{E}(I_t) = 0$ and

$$\mathrm{Var}(I_t) = \int_0^t g^2(s)\,\mathrm{d}s.$$

As in the previous section, however, examining the Riemann sum

$$I_t^{(n)} = \sum_{j=1}^{n} g(t_{j-1})(B_{t_j} - B_{t_{j-1}})$$

suggests that we can determine more than just the mean and variance of $I_t^{(n)}$. Since disjoint Brownian increments are independent and normally distributed, and since $I_t^{(n)}$ is a sum of disjoint Brownian increments, we conclude that $I_t^{(n)}$ is normally distributed. In fact, combined with our earlier calculations, we see from Exercise 3.12 that

$$I_t^{(n)} \sim \mathcal{N}\left(0, \sum_{j=1}^{n} g^2(t_{j-1})(t_j - t_{j-1})\right).$$

It now follows from Theorem 3.24 that $I_t^{(n)}$ converges in distribution to the random variable I_t where

$$I_t \sim \mathcal{N}\left(0, \int_0^t g^2(s)\,\mathrm{d}s\right)$$

since the limit in distribution of normal random variables whose means and variances converge must itself be normal. Hence, we define

$$\int_0^t g(s)\,\mathrm{d}B_s$$

to be this limit I_t so that

$$I_t = \int_0^t g(s)\,\mathrm{d}B_s \sim \mathcal{N}\left(0, \int_0^t g^2(s)\,\mathrm{d}s\right).$$

Definition 8.1. Suppose that $g : [0,\infty) \to \mathbb{R}$ is a bounded, piecewise continuous function in $L^2([0,\infty))$. The *Wiener integral* of g with respect to Brownian motion $\{B_t, t \geq 0\}$, written

$$\int_0^t g(s)\,\mathrm{d}B_s,$$

is a random variable which has a

$$\mathcal{N}\left(0, \int_0^t g^2(s)\,\mathrm{d}s\right)$$

distribution.

Remark. We have taken the approach of defining the Wiener integral in a distributional sense. It is possible, with a lot more technical machinery, to define it as the L^2 limit of a sequence of random variables. In the case of a random g, however, in order to the define the Itô integral of g with respect to Brownian motion, we will need to follow the L^2 approach. Furthermore, we will see that the Wiener integral is actually a special case of the Itô integral. Thus, it seems pedagogically more appropriate to define the Wiener integral in the distributional sense since this is a much simpler construction and, arguably, more intuitive.