Statistics 354 (Fall 2018)
Review of Linear Algebra

Suppose that A is the symmetric matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$

Determine the eigenvalues and eigenvectors of A.

Recall that a real number λ is an eigenvalue of A if $A\mathbf{v} = \lambda\mathbf{v}$ for some vector $\mathbf{v} \neq 0$. We call $\mathbf{v}$ an eigenvector (corresponding to the eigenvalue λ) of A. Note that if $\mathbf{v}$ is an eigenvector of A, then so too is $\alpha\mathbf{v}$ for any non-zero real number α. The non-zero vector $\mathbf{v}$ is a solution of the equation $A\mathbf{v} = \lambda\mathbf{v}$ if and only if $\mathbf{v}$ is also a solution of the equation $(A - \lambda I)\mathbf{v} = 0$. The equation $(A - \lambda I)\mathbf{v} = 0$ has a non-zero solution if and only if the matrix $A - \lambda I$ is singular (non-invertible). The matrix $A - \lambda I$ is invertible if and only if $\det[A - \lambda I] \neq 0$. Therefore, in order to find the eigenvalues of A, we need to find those values of λ such that $\det[A - \lambda I] = 0$. (The equation $\det[A - \lambda I] = 0$ is also known as the *characteristic equation* of the matrix A.) Therefore, we consider

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{bmatrix}.$$

Since

$$\begin{aligned} \det \begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{bmatrix} &= (1-\lambda)(2-\lambda)(3-\lambda) - (1-\lambda) - (3-\lambda) \\ &= 2 - 9\lambda + 6\lambda^2 - \lambda^3 \\ &= (2-\lambda)(\lambda^2 - 4\lambda + 1) \\ &= (2-\lambda)(\lambda - 2 - \sqrt{3})(\lambda - 2 + \sqrt{3}) \end{aligned}$$

we conclude that there are 3 eigenvalues, namely

$$\lambda_1 = 2, \quad \lambda_2 = 2 - \sqrt{3}, \quad \lambda_3 = 2 + \sqrt{3}.$$

If λ is an eigenvalue of A, then we can determine the corresponding eigenvectors by row reduction. That is, for $\lambda_1 = 2$,

$$[A - \lambda_1 I \,|\, 0] = \left[\begin{array}{ccc|c} -1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

For $\lambda_2 = 2 - \sqrt{3}$,

$$[A - \lambda_2 I \,|\, 0] = \left[\begin{array}{ccc|c} -1+\sqrt{3} & -1 & 0 & 0 \\ -1 & \sqrt{3} & 1 & 0 \\ 0 & 1 & 1+\sqrt{3} & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2+\sqrt{3} & 0 \\ 0 & 1 & 1+\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

For $\lambda_3 = 2 + \sqrt{3}$,

$$[A - \lambda_3 I \,|\, 0] = \left[\begin{array}{ccc|c} -1-\sqrt{3} & -1 & 0 & 0 \\ -1 & -\sqrt{3} & 1 & 0 \\ 0 & 1 & 1-\sqrt{3} & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2-\sqrt{3} & 0 \\ 0 & 1 & 1-\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Since the eigenvectors corresponding to a given eigenvalue λ lie in the nullspace of $[A - \lambda I]$, we conclude that a basis for the eigenspace corresponding to λ_1 is

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

a basis for the eigenspace corresponding to λ_2 is

$$\mathbf{v}_2 = \begin{bmatrix} -2-\sqrt{3} \\ -1-\sqrt{3} \\ 1 \end{bmatrix},$$

and a basis for the eigenspace corresponding to λ_3 is

$$\mathbf{v}_3 = \begin{bmatrix} -2+\sqrt{3} \\ -1+\sqrt{3} \\ 1 \end{bmatrix}.$$

Diagonalize A

Since the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 2 - \sqrt{3}$, and $\lambda_3 = 2 + \sqrt{3}$, we conclude that

$$D = \text{diag}[\lambda_1, \lambda_2, \lambda_3] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2-\sqrt{3} & 0 \\ 0 & 0 & 2+\sqrt{3} \end{bmatrix}.$$

The orthogonal matrix C is given by

$$C = \left[\frac{\mathbf{v}_1}{||\mathbf{v}_1||} \; \frac{\mathbf{v}_2}{||\mathbf{v}_2||} \; \frac{\mathbf{v}_3}{||\mathbf{v}_3||} \right].$$

(That is, the ith column of C contains the elements of the normalized eigenvector corresponding to λ_i, which appears as the (i, i) entry of D.) Thus,

$$C = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-2-\sqrt{3}}{3+\sqrt{3}} & \frac{-2+\sqrt{3}}{3-\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{-1-\sqrt{3}}{3+\sqrt{3}} & \frac{-1+\sqrt{3}}{3-\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{3+\sqrt{3}} & \frac{1}{3-\sqrt{3}} \end{bmatrix}.$$

One can easily check that

$$
\begin{aligned}
C'AC &= \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-2-\sqrt{3}}{3+\sqrt{3}} & \frac{-1-\sqrt{3}}{3+\sqrt{3}} & \frac{1}{3+\sqrt{3}} \\ \frac{-2+\sqrt{3}}{3-\sqrt{3}} & \frac{-1+\sqrt{3}}{3-\sqrt{3}} & \frac{1}{3-\sqrt{3}} \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-2-\sqrt{3}}{3+\sqrt{3}} & \frac{-2+\sqrt{3}}{3-\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{-1-\sqrt{3}}{3+\sqrt{3}} & \frac{-1+\sqrt{3}}{3-\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{3+\sqrt{3}} & \frac{1}{3-\sqrt{3}} \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2-\sqrt{3} & 0 \\ 0 & 0 & 2+\sqrt{3} \end{bmatrix} \\
&= D
\end{aligned}
$$

or, equivalently, that $A = CDC'$.

Calculate $\det A$

Solution 1. Since $\lambda_1 = 2$, $\lambda_2 = 2-\sqrt{3}$, and $\lambda_3 = 2+\sqrt{3}$, we conclude that

$$\det[A] = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 2(2-\sqrt{3})(2+\sqrt{3}) = 2.$$

Solution 2. The determinant of A can be calculated directly, namely

$$
\begin{aligned}
\det[A] = \det &\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} \\
&= 1 \cdot 2 \cdot 3 + (-1) \cdot 1 \cdot 0 + 0 \cdot (-1) \cdot 1 - 0 \cdot 2 \cdot 0 - 1 \cdot 1 \cdot 1 - 3 \cdot (-1) \cdot (-1) \\
&= 6 - 1 - 3 = 2.
\end{aligned}
$$

Determine the quadratic form Q associated with A

Suppose that

$$\overline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

is a column vector in $\mathbb{R}^3$. By definition, the quadratic form Q associated with A is given by

$$
\begin{aligned}
Q(\overline{x}) = \overline{x}' A \overline{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} &\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&= \begin{bmatrix} x_1 - x_2 & -x_1 + 2x_2 + x_3 & x_2 + 3x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&= x_1^2 - x_1x_2 - x_1x_2 + 2x_2^2 + x_2x_3 + x_2x_3 + 3x_3^2 \\
&= x_1^2 - 2x_1x_2 + 2x_2^2 + 2x_2x_3 + 3x_3^2.
\end{aligned}
$$

Determine if Q is either positive definite or non-negative definite

Solution 1. Since all the eigenvalues of A, namely $\lambda_1 = 2$, $\lambda_2 = 2 - \sqrt{3}$, and $\lambda_3 = 2 + \sqrt{3}$, are strictly positive, we conclude that A is positive definite.

Solution 2. Since

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

the three upper left block matrices are

$$A_1 = \begin{bmatrix} 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \text{ and } \quad A_3 = A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$

We compute $\det[A_1] = 1$, $\det[A_2] = 1$, and $\det[A_3] = \det[A] = 2$. Therefore, we conclude that the quadratic form Q associated with A is positive definite since each upper left block matrix has a positive determinant.