Statistics 354 (Fall 2018)

Joint distribution of $\hat{\beta}_0$, $\hat{\beta}_1$ in the simple linear regression model

Consider the simple linear regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$, $i = 1, \ldots, n$, where $\epsilon_1, \ldots, \epsilon_n$ are iid $\mathcal{N}(0, \sigma^2)$. Let $\hat{\beta}_0$, $\hat{\beta}_1$ denote the least squares estimators of β_0, β_1, respectively. We can write the simple linear regression model as $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}.$$

It was shown in class that

$$\hat{\boldsymbol{\beta}} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} \sim \mathcal{N}(\boldsymbol{\beta}, \sigma^2(\mathbf{X}'\mathbf{X})^{-1}).$$

We now find

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}.$$

so that

$$(\mathbf{X}'\mathbf{X})^{-1} = \frac{1}{n\left(\sum x_i^2\right) - \left(\sum x_i\right)^2} \begin{bmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{bmatrix}.$$

Next we observe that

$$s_{xx} = \sum (x_i - \overline{x})^2 = \left(\sum x_i^2\right) - n\overline{x}^2$$

which implies

$$n\left(\sum x_i^2\right) - \left(\sum x_i\right)^2 = n\left(\sum x_i^2\right) - n^2\overline{x}^2 = n s_{xx}.$$

Moreover,

$$\sum x_i^2 = s_{xx} + n\overline{x}^2$$

so that

$$(\mathbf{X}'\mathbf{X})^{-1} = \frac{1}{n s_{xx}} \begin{bmatrix} s_{xx} + n\overline{x}^2 & -n\overline{x} \\ -n\overline{x} & n \end{bmatrix} = \begin{bmatrix} \frac{1}{n} + \frac{\overline{x}^2}{s_{xx}} & -\frac{\overline{x}}{s_{xx}} \\ -\frac{\overline{x}}{s_{xx}} & \frac{1}{s_{xx}} \end{bmatrix}.$$

Therefore, we conclude that

$$\hat{\boldsymbol{\beta}} \sim \mathcal{N}\left(\boldsymbol{\beta}, \begin{bmatrix} \sigma^2\left(\frac{1}{n} + \frac{\overline{x}^2}{s_{xx}}\right) & -\frac{\sigma^2\overline{x}}{s_{xx}} \\ -\frac{\sigma^2\overline{x}}{s_{xx}} & \frac{\sigma^2}{s_{xx}} \end{bmatrix}\right).$$

In particular,

$$\hat{\beta}_0 \sim \mathcal{N}\left(\beta_0,\ \sigma^2\left(\frac{1}{n} + \frac{\overline{x}^2}{s_x x}\right)\right) \quad \text{and} \quad \hat{\beta}_1 \sim \mathcal{N}\left(\beta_1, \frac{\sigma^2}{s_{xx}}\right)$$

as derived earlier in class.