Statistics 354 Fall 2018 Midterm – Solutions

1. We showed in class that

$$\hat{\boldsymbol{\beta}} \sim \mathcal{N}\left(\boldsymbol{\beta}, \begin{bmatrix} \sigma^2\left(\frac{1}{n}+\frac{\overline{x}^2}{s_{xx}}\right) & -\frac{\sigma^2\overline{x}}{s_{xx}} \\ -\frac{\sigma^2\overline{x}}{s_{xx}} & \frac{\sigma^2}{s_{xx}} \end{bmatrix}\right).$$

Therefore, $\hat{\mu}_0 = \hat{\beta}_0 + \hat{\beta}_1 x_0$ has a normal distribution since it is a linear combination of the components of a multivariate normal, and so all that is left is to compute $\mathbb{E}(\hat{\mu}_0)$ and $\operatorname{Var}(\hat{\mu}_0)$. Since $\mathbb{E}(\hat{\mu}_0) = \mathbb{E}(\hat{\beta}_0 + \hat{\beta}_1 x_0) = \mathbb{E}(\hat{\beta}_0) + x_0\mathbb{E}(\hat{\beta}_1) = \beta_0 + \beta_1 x_0 = \mu_0$ and

$$\begin{aligned}\operatorname{Var}(\hat{\mu}_0) = \operatorname{Var}(\hat{\beta}_0 + \hat{\beta}_1 x_0) &= \operatorname{Var}(\hat{\beta}_0) + x_0^2\operatorname{Var}(\hat{\beta}_1) + 2x_0\operatorname{Cov}(\hat{\beta}_0, \beta_1) \\ &= \sigma^2\left(\frac{1}{n}+\frac{\overline{x}^2}{s_{xx}}\right) + \frac{\sigma^2 x_0^2}{s_{xx}} - 2\frac{\sigma^2 x_0 \overline{x}}{s_{xx}} \\ &= \sigma^2\left(\frac{1}{n}+\frac{(x_0-\overline{x})^2}{s_{xx}}\right),\end{aligned}$$

we conclude

$$\hat{\mu}_0 \sim \mathcal{N}\left(\mu_0,\, \sigma^2\left(\frac{1}{n}+\frac{(x_0-\overline{x})^2}{s_{xx}}\right)\right).$$

2. If we define

$$\boldsymbol{\beta} = \begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix}y_1\\ \vdots \\ y_n\end{bmatrix}, \quad \boldsymbol{\epsilon} = \begin{bmatrix}\epsilon_1\\ \vdots \\ \epsilon_n\end{bmatrix}, \quad \text{and} \quad \mathbf{X} = \begin{bmatrix}x_{11} & x_{12}\\ \vdots & \vdots \\ x_{n1} & x_{n2}\end{bmatrix},$$

then $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ as required. Since

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix}x_{11} & \dots & x_{n1}\\ x_{12} & \dots & x_{n2}\end{bmatrix}\begin{bmatrix}x_{11} & x_{12}\\ \vdots & \vdots \\ x_{n1} & x_{n2}\end{bmatrix} = \begin{bmatrix}\sum x_{i1}^2 & \sum x_{i1}x_{i2}\\ \sum x_{i1}x_{i2} & \sum x_{i2}^2\end{bmatrix}$$

so that

$$(\mathbf{X}'\mathbf{X})^{-1} = \frac{1}{\left(\sum x_{i1}^2\right)\left(\sum x_{i2}^2\right) - \left(\sum x_{i1}\sum x_{i2}\right)^2}\begin{bmatrix}\sum x_{i2}^2 & -\sum x_{i1}x_{i2}\\ \sum x_{i1}x_{i2} & \sum x_{i1}^2\end{bmatrix},$$

we conclude

$$\hat{\beta}_1 \sim \mathcal{N}\left(\beta_1, \frac{\sigma^2\sum x_{i2}^2}{\left(\sum x_{i1}^2\right)\left(\sum x_{i2}^2\right) - \left(\sum x_{i1}x_{i2}\right)^2}\right), \quad \hat{\beta}_2 \sim \mathcal{N}\left(\beta_2, \frac{\sigma^2\sum x_{i1}^2}{\left(\sum x_{i1}^2\right)\left(\sum x_{i2}^2\right) - \left(\sum x_{i1}x_{i2}\right)^2}\right),$$

and

$$\operatorname{Cov}(\hat{\beta}_1, \hat{\beta}_2) = -\frac{\sigma^2\sum x_{i1}x_{i2}}{\left(\sum x_{i1}^2\right)\left(\sum x_{i2}^2\right) - \left(\sum x_{i1}x_{i2}\right)^2}.$$

3. (a) We have $\hat{\boldsymbol{\mu}}'\mathbf{e} = (H\mathbf{y})'(1 - H)\mathbf{y} = \mathbf{y}'H'(I - H)\mathbf{y} = [0]$ since $H' = H$ and $H^2 = H$.

3. (b) Since $\mathbf{y}$ has a multivariate normal distribution, we know that the random vector $[\boldsymbol{\mu}, \mathbf{e}]'$ has a multivariate normal distribution. The previous result shows that $\mathrm{Cov}(\hat{\mu}_i, e_j) = 0$ for all i and j which implies that $\hat{\mu}_i$ and e_j are independent for all i and j since the components of a multivariate normal are independent if and only if they are uncorrelated. Thus, $\boldsymbol{\mu}$ and $\mathbf{e}$ are independent.