Stat 354 Fall 2018
Solutions to Assignment #2

1. We know that $y_1 \ldots, y_n$ are iid with $y_i \sim \mathcal{N}(\beta_0 + \beta_1 x_i, \sigma^2)$. Hence, the fact that $\hat{\beta}_0$ and $\hat{\beta}_1$ are each linear combinations of $y_1, \ldots, y_n$ (in other words, $\hat{\beta}_0$ and $\hat{\beta}_1$ are linear combinations of independent normal random variables), namely

$$\hat{\beta}_0 = \sum_{i=1}^{n} k_i y_i \quad \text{where} \quad k_i = \frac{1}{n} - \frac{\overline{x}(x_i - \overline{x})}{s_{xx}}$$

and

$$\hat{\beta}_1 = \sum_{i=1}^{n} c_i y_i \quad \text{where} \quad c_i = \frac{x_i - \overline{x}}{s_{xx}},$$

implies that

$$\hat{\boldsymbol{\beta}} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix}$$

has a multivariate normal distribution. We showed in class that

$$\hat{\beta}_0 \sim \mathcal{N}\left(\beta_0, \sigma^2 \left(\frac{1}{n} + \frac{\overline{x}^2}{s_{xx}}\right)\right)$$

and

$$\hat{\beta}_1 \sim \mathcal{N}\left(\beta_1, \frac{\sigma^2}{s_{xx}}\right).$$

Therefore, the final step is to determine $\operatorname{Cov}(\hat{\beta}_0, \hat{\beta}_1)$. We begin by observing that

$$\operatorname{Cov}(\hat{\beta}_0, \hat{\beta}_1) = \operatorname{Cov}\left(\sum_{i=1}^{n} k_i y_i, \sum_{i=1}^{n} c_i y_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i k_j \operatorname{Cov}(y_i, y_j) = \sum_{i=1}^{n} c_i k_i \operatorname{Var}(y_i) = \sigma^2 \sum_{i=1}^{n} c_i k_i$$

since $\operatorname{Cov}(y_i, y_j) = 0$ if $i \neq j$ and $\operatorname{Var}(y_i) = \sigma^2$. We now observe that

$$\sum_{i=1}^{n} c_i k_i = \sum_{i=1}^{n} c_i k_i = \sum_{i=1}^{n} \left(\frac{1}{n} - \frac{\overline{x}(x_i - \overline{x})}{s_{xx}}\right)\left(\frac{x_i - \overline{x}}{s_{xx}}\right) = \frac{1}{n s_{xx}} \sum_{i=1}^{n} (x_i - \overline{x}) - \frac{\overline{x}}{s_{xx}^2} \sum_{i=1}^{n} (x_i - \overline{x})^2 = -\frac{\overline{x}}{s_{xx}}$$

so that

$$\operatorname{Cov}(\hat{\beta}_0, \hat{\beta}_1) = -\frac{\sigma^2 \overline{x}}{s_{xx}}.$$

Putting everything together implies

$$\hat{\boldsymbol{\beta}} \sim \mathcal{N}\left(\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}, \begin{bmatrix} \sigma^2 \left(\frac{1}{n} + \frac{\overline{x}^2}{s_{xx}}\right) & -\frac{\sigma^2 \overline{x}}{s_{xx}} \\ -\frac{\sigma^2 \overline{x}}{s_{xx}} & \frac{\sigma^2}{s_{xx}} \end{bmatrix}\right).$$

(3.1) If

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 2 & 2 \\ 2 & 1 & 4 \end{bmatrix},$$

then

$$A' = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix} \quad \text{and} \quad A'A = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 3 & 2 & 2 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 17 & 8 & 16 \\ 8 & 5 & 8 \\ 16 & 8 & 21 \end{bmatrix}.$$

We find

$$\mathrm{tr}(A) = 2 + 2 + 4 = 8 \quad \text{and} \quad \mathrm{tr}(A'A) = 17 + 5 + 21 = 43$$

as well as

$$\det(A) = 2 \cdot 2 \cdot 4 + 0 + 1 \cdot 3 \cdot 1 - 1 \cdot 2 \cdot 2 - 0 - 2 \cdot 1 \cdot 2 = 11$$

and

$$\det(A'A) = 17 \cdot 5 \cdot 21 + 8 \cdot 8 \cdot 16 + 16 \cdot 8 \cdot 8 - 16 \cdot 5 \cdot 16 - 8 \cdot 8 \cdot 17 - 21 \cdot 8 \cdot 8 = 121.$$

(3.2) If

$$\mathbf{X} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & +1 & -1 \\ 1 & -1 & +1 \\ 1 & +1 & +1 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 4 \\ 3 \\ 5 \\ 7 \end{bmatrix},$$

then

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & +1 & -1 & +1 \\ -1 & -1 & +1 & +1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 1 & +1 & -1 \\ 1 & -1 & +1 \\ 1 & +1 & +1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

and

$$(\mathbf{X}'\mathbf{X})^{-1} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}.$$

We also find

$$\mathbf{X}'\mathbf{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & +1 & -1 & +1 \\ -1 & -1 & +1 & +1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 19 \\ 1 \\ 5 \end{bmatrix}$$

and so

$$(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} 19 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 19/4 \\ 1/4 \\ 5/4 \end{bmatrix}.$$

Note that both $\mathbf{X}'\mathbf{X}$ and $(\mathbf{X}'\mathbf{X})^{-1}$ are diagonal matrices.

(3.3) If

$$\mathbf{X} = \begin{bmatrix} 1 & x_{11} & x_{12} \\ 1 & x_{21} & x_{22} \\ 1 & x_{31} & x_{32} \\ 1 & x_{41} & x_{42} \\ 1 & x_{51} & x_{52} \end{bmatrix}$$

then

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ x_{11} & x_{21} & x_{31} & x_{41} & x_{51} \\ x_{12} & x_{22} & x_{32} & x_{42} & x_{52} \end{bmatrix} \begin{bmatrix} 1 & x_{11} & x_{12} \\ 1 & x_{21} & x_{22} \\ 1 & x_{31} & x_{32} \\ 1 & x_{41} & x_{42} \\ 1 & x_{51} & x_{52} \end{bmatrix} = \begin{bmatrix} 1 & \sum x_{i1} & \sum x_{i2} \\ \sum x_{i1} & \sum x_{i1}^2 & \sum x_{i1}x_{i2} \\ \sum x_{i2} & \sum x_{i1}x_{i2} & \sum x_{i2}^2 \end{bmatrix}.$$

(3.4) If

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

then $\det(A) = (2)(2) - (-1)(-1) = 3$ and

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}.$$

We now find

$$A - \lambda I = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2/-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix}$$

so that

$$\det(A - \lambda I) = (2-\lambda)^2 - (1)^2 = (1-\lambda)(3-\lambda).$$

(The difference of perfect squares is easy to factor.) Thus, there are two eigenvalues, namely

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = 3.$$

The corresponding eigenvectors are found by row reduction. For $\lambda_1 = 1$,

$$[A - \lambda_1 I \,|\, 0] = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

and for $\lambda_2 = 3$,

$$[A - \lambda_2 I \,|\, 0] = \left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Since the eigenvectors corresponding to a given eigenvalue λ lie in the nullspace of $[A - \lambda I]$, we conclude that a basis for the eigenspace corresponding to λ_1 is

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and a basis for the eigenspace corresponding to λ_2 is

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The normalized eigenvectors are therefore

$$\mathbf{p}_1 = \frac{\mathbf{v}_1}{||\mathbf{v}_1||} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \text{and} \quad \mathbf{p}_2 = \frac{\mathbf{v}_2}{||\mathbf{v}_2||} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

(3.6) If

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

then $\det(A) = 3$ and

$$A^{-1} = \begin{bmatrix} 4/3 & -1/3 & -4/3 \\ -1/3 & 1/3 & 1/3 \\ -4/3 & 1/3 & 7/3 \end{bmatrix}.$$

(3.12) Let $\mathbf{z} = [z_1, z_2, z_3]'$ be the random vector with mean vector $\boldsymbol{\mu}$ and covariance matrix V. If $\mathbf{y} = [y_1, y_2, y_3]'$ and

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{bmatrix}$$

then $\mathbf{z} = A\mathbf{y} - [0, 0, 7]'$. Hence, the mean vector of $\mathbf{y}$ is

$$\mathbb{E}(\mathbf{y}) = A\boldsymbol{\mu} - \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix}.$$

The covariance matrix of $\mathbf{y}$ is

$$V(\mathbf{y}) = AVA' = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 6 & 17 \\ 6 & 6 & 12 \\ 17 & 12 & 29 \end{bmatrix}.$$

Let $\mathbf{w} = [1, 1, 1]'$ so that

$$y = \frac{1}{3}(y_1 + y_2 + y_3) = \frac{1}{3}\mathbf{w}'\mathbf{y}.$$

Therefore,

$$\mathbb{E}(y) = \frac{1}{3}\mathbb{E}(\mathbf{w}'\mathbf{y}) = \frac{1}{3}\mathbf{w}'\mathbb{E}(\mathbf{y}) = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix} = \frac{7}{3}$$

and

$$V(y) = \frac{1}{9}V(\mathbf{w}'\mathbf{y}) = \frac{1}{9}\mathbf{w}'V(\mathbf{y})\mathbf{w} = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 11 & 6 & 17 \\ 6 & 6 & 12 \\ 17 & 12 & 29 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{116}{9}.$$

(3.15) We begin by noting that

$$A - \lambda I = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & \rho \\ \rho & 1-\lambda \end{bmatrix}$$

so that

$$\det(A - \lambda I) = (1-\lambda)^2 - \rho^2 = (1 - \lambda + \rho)(1 - \lambda - \rho).$$

(The difference of perfect squares is easy to factor.) Thus, there are two eigenvalues, namely

$$\lambda_1 = 1 + \rho \quad \text{and} \quad \lambda_2 = 1 - \rho.$$

The corresponding eigenvectors are found by row reduction. For $\lambda_1 = 1 + \rho$,

$$[A - \lambda_1 I \,|\, 0] = \left[\begin{array}{cc|c} -\rho & \rho & 0 \\ \rho & -\rho & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

and for $\lambda_2 = 1 - \rho$,

$$[A - \lambda_2 I \,|\, 0] = \left[\begin{array}{cc|c} \rho & \rho & 0 \\ \rho & \rho & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Since the eigenvectors corresponding to a given eigenvalue λ lie in the nullspace of $[A - \lambda I]$, we conclude that a basis for the eigenspace corresponding to λ_1 is

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and a basis for the eigenspace corresponding to λ_2 is

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The normalized eigenvectors are therefore

$$\mathbf{p}_1 = \frac{\mathbf{v}_1}{||\mathbf{v}_1||} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \text{and} \quad \mathbf{p}_2 = \frac{\mathbf{v}_2}{||\mathbf{v}_2||} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$