Statistics 352 Winter 2008 Midterm # 2 — Solutions

1. (a) The posterior density is given by

$$f(\theta|y=1) = C\frac{\theta^4 e^{\theta/2}}{e^\theta - 1}$$

where C satisfies

$$\frac{1}{C} = \int_0^\infty \frac{\theta^4 e^{\theta/2}}{e^\theta - 1}\,\mathrm{d}\theta.$$

Using Maple we can find the value of C as follows.

```
> Int(x^4*exp(-x/2)/((exp(x)-1)),x=0..infinity);
> evalf(%);
                         3.474249827
```

Thus,

$$f(\theta|y=1) = \frac{\theta^4 e^{\theta/2}}{3.474249827(e^\theta - 1)}, \quad \theta > 0.$$

1. (b) The posterior probability is given by

$$P\{0<\theta<1|y=1\} = \int_0^1 f(\theta|y=1)\,\mathrm{d}\theta = \int_0^1 \frac{\theta^4 e^{\theta/2}}{3.474249827(e^\theta - 1)}\,\mathrm{d}\theta.$$

The value of this integral can be found using MAPLE as follows.

```
> Int(x^4*exp(-x/2)/(3.474249827*(exp(x)-1)),x=0..1);
> evalf(%);
                       0.03195496305
```

Thus, the required posterior probability is $P\{0<\theta<1|y=1\} = 0.03195496305$.

1. (c) An equal-tailed 90% Bayesian credible interval is given by $[L, R]$ where L and R satisfy

$$\int_0^L f(\theta|y=1)\,\mathrm{d}\theta = \int_0^L \frac{\theta^4 e^{\theta/2}}{3.474249827(e^\theta - 1)}\,\mathrm{d}\theta = 0.05$$

and

$$\int_0^R f(\theta|y=1)\,\mathrm{d}\theta = \int_0^R \frac{\theta^4 e^{\theta/2}}{3.474249827(e^\theta - 1)}\,\mathrm{d}\theta = 0.95.$$

Using MAPLE we can find the values of L and R as follows.

```
> Int(x^4*exp(-x/2)/(3.474249827*(exp(x)-1)),x=0..1.1544657838);
> evalf(%);
                       0.05000000000

> Int(x^4*exp(-x/2)/(3.474249827*(exp(x)-1)),x=0..6.001314396);
> evalf(%);
                        0.9500000000
```

Thus, the required 90% credible interval is $[1.1544657838, 6.001314396]$.

1. (d) The following code runs the envelope method to simulate from the posterior density $f(\theta|y=1)$. The output is a histogram of sampled values.

```
> Y=0
> N=10000
> y <- rexp(N,0.5)
> u <- runif(N,0,25*0.5*exp(-y/2))
> f <- y^4*exp(-y/2)/(3.474249827*(exp(y)-1))
> m=pmax(f-u,0)
> for(i in 1:N) ifelse(m[i]==0, Y[i]<-NA,Y[i]<-y[i])
> tmp <-na.omit(Y)
> X=0
> for (i in 1:length(tmp)) X[i]<-tmp[i]
> hist(X)
```

2. (a) The following program uses a Gibbs sampler to simulate a pair (X, Y) from the density $f(x, y)$.

```
gibbs <- function(numsteps,x0)
{
mat <- matrix(ncol=2,nrow=numsteps)
x<- x0
y <- rnorm(1,1,abs(x))
mat[1,] <- c(x,y)
for (i in 2:numsteps) {
x <- rnorm(1,1,abs(y))
y <- rnorm(1,1,abs(x))
mat[i,] <- c(x,y)
}
mat
}
```

Running `gibbs(31,1)` gives

```
[31,]  3.77437736  4.41922728
```

and so our sampled values are $X = 3.77437736$, $Y = 4.41922728$.

2. (b) The following program creates a loop to run the previous Gibbs sampler $N = 1000$ times.

```
multigibbs <-function(iterations,x0, numsteps)
{
Z <- matrix(ncol=2,nrow=iterations)
for (i in 1:iterations) {
a=gibbs(numsteps,x0)
Z[i,] <- c(a[numsteps,])
}
Z
}
```

Running `multigibbs(1000,1,31)` gives the required sample of size 1000. In order to estimate $\mathbb{E}(X)$ and $\mathbb{E}(Y)$ we use the following code.

```
> A = multigibbs(1000,1,31)
> mean(A[,1])
[1] 1.065511
> mean(A[,2])
[1] 1.039188
```

In particular, $\mathbb{E}(X) \approx 1.065511$ and $\mathbb{E}(Y) \approx 1.039188$.

It is a fact that in this example $f(x,y)$ is not multivariate normal even though the conditional densities are normal.

3. Using Maple to compute $\mathbf{P}^{23}$ gives

$$\mathbf{P}^{23} = \begin{pmatrix} 0.2820512821 & 0.4102564103 & 0.3076923077 \\ 0.2820512821 & 0.4102564103 & 0.3076923077 \\ 0.2820512821 & 0.4102564103 & 0.3076923077 \end{pmatrix}.$$

Thus, the long run probability that the Markov chain is in state 1 is approximately 0.282.

Solving $\bar{\pi}\mathbf{P} = \bar{\pi}$ where $\bar{\pi} = (\pi_1, \pi_2, \pi_3)$ gives the following system of equations:

$$\begin{aligned} 0.2\pi_1 &+ 0.1\pi_2 + 0.6\pi_3 = \pi_1 \\ 0.4\pi_1 &+ 0.5\pi_2 + 0.3\pi_3 = \pi_2 \\ 0.4\pi_1 &+ 0.4\pi_2 + 0.1\pi_3 = \pi_3 \end{aligned}$$

Row reducing yields

$$-3\pi_2 + 4\pi_3 = 0 \quad \text{and} \quad -16\pi_1 + 11\pi_2 = 0$$

and so using the fact that $\pi_1 + \pi_2 + \pi_3 = 1$ we conclude

$$\frac{11}{16}\pi_2 + \pi_2 + \frac{3}{4}\pi_2 = 1.$$

Thus,

$$\bar{\pi} = (\pi_1, \pi_2, \pi_3) = \left(\frac{11}{39}, \frac{16}{39}, \frac{12}{39}\right)$$

and so we conclude that the long run probability that the Markov chain is in state 1 is $\frac{11}{39} \approx 0.282$.