Statistics 351 Fall 2015 Midterm #2 – Solutions

1. (a) The joint density for $X_{(1)}, X_{(2)}, X_{(3)}$ is

$$f_{X_{(1)},X_{(2)},X_{(3)}}(y_1, y_2, y_3) = 3!, \quad 0 \le y_1 \le y_2 \le y_3 \le 1.$$

so that the joint density for $X_{(1)}, X_{(2)}$ is

$$f_{X_{(1)},X_{(2)}}(y_1, y_2) = \int_{-\infty}^{\infty} f_{X_{(1)},X_{(2)},X_{(3)}}(y_1, y_2, y_3)\,\mathrm{d}y_3 = \int_{y_2}^{1} 6\,\mathrm{d}y_3 = 6(1-y_2)$$

provided that $0 \le y_1 \le y_2 \le 1$.

1. (b) We find

$$\begin{aligned}
P(X_{(2)} < 2X_{(1)}) &= \iint_{\{y_2<2y_1\}} f_{X_{(1)},X_{(2)}}(y_1, y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2 = \int_0^1 \int_{y_2/2}^{y_2} 6(1-y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2 \\
&= \int_0^1 6(1-y_2)\left[\int_{y_2/2}^{y_2} 1\,\mathrm{d}y_1\right]\mathrm{d}y_2 = \int_0^1 6(1-y_2)(y_2 - y_2/2)\,\mathrm{d}y_2 \\
&= \int_0^1 3y_2(1-y_2)\,\mathrm{d}y_2 = \left[\frac{3}{2}y_2^2 - y_2^3\right]_{y_2=0}^{y_2=1} \\
&= \frac{1}{2}.
\end{aligned}$$

2. Since X_1, X_2 are iid Exp(1), the joint density of $X_{(1)}$, $X_{(2)}$ is

$$f_{X_{(1)},X_{(2)}}(y_1, y_2) = 2f(y_1)f(y_2) = 2e^{-y_1-y_2}, \quad 0 < y_1 \le y_2 < \infty.$$

If $U = X_{(2)} - X_{(1)}$ and $V = X_{(1)}$, then solving for $X_{(1)}$ and $X_{(2)}$ implies $X_{(1)} = V$ and $X_{(2)} = U + V$. The Jacobian of this transformation is

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial u} & \frac{\partial y_1}{\partial v} \\ \frac{\partial y_2}{\partial u} & \frac{\partial y_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$$

implying that $f_{U,V}(u, v) = f_{X_{(1)},X_{(2)}}(v, u+v) \cdot |J| = 2e^{-u-(u+v)} = 2e^{-2u}e^{-v}$ provided $u > 0$, $v > 0$. (Note that $u > 0$ and $v > 0$ implies that necessarily $v < u + v$.) Hence, we see that if $f_U(u) = 2e^{-2u}$, $u > 0$, which is the density of an Exp(1/2) random variable, and if $f_V(v) = e^{-v}$, $v > 0$, which is the density of an Exp(1) random variable, then $f_{U,V}(u, v) = f_U(u)f_V(v)$ implying that U and V are independent.

3. If $\mathbf{X} = (X_1, X_2)'$, then since the components of $\mathbf{X}$ are independent and normally distributed, $\mathbf{X}$ has a multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Lambda}$ where

$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let

$$B = \begin{bmatrix} 1 & -2 \\ 2 & 3 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

so that $\mathbf{Y} = B\mathbf{X} + \mathbf{b}$. By Theorem 5.3.1, $\mathbf{Y}$ is MVN with mean

$$B\boldsymbol{\mu} + \mathbf{b} = \begin{bmatrix} 1 & -2 \\ 2 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

and covariance matrix

$$B\boldsymbol{\Lambda}B' = \begin{bmatrix} 1 & -2 \\ 2 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ -2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 13 & 3 \\ -2 & 3 & 1 \end{bmatrix}.$$

4. (a) Since $\det[\boldsymbol{\Lambda}] = 1 - \rho^2$ and

$$\Lambda^{-1} = \frac{1}{1-\rho^2} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}$$

we conclude that the density for $\mathbf{X}$ is

$$\begin{aligned} f_{\mathbf{X}}(x,y) &= \frac{1}{2\pi\sqrt{\det[\boldsymbol{\Lambda}]}} \exp\left\{ -\frac{1}{2} \begin{bmatrix} x & y \end{bmatrix} \boldsymbol{\Lambda}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} \right\} \\ &= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)} (x^2 - 2\rho xy + y^2) \right\} \end{aligned}$$

4. (b) If $\rho = 0$, then

$$f_{\mathbf{X}}(x,y) = \frac{1}{2\pi} \exp\left\{ -\frac{1}{2}(x^2+y^2) \right\} = \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{x^2}{2} \right\} \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{y^2}{2} \right\} = f_X(x) f_Y(y)$$

showing that the joint density for $(X,Y)'$ can be factored into the product of the densities for $X \in N(0,1)$ and $Y \in N(0,1)$. This proves that X and Y are independent.