Statistics 351 Fall 2009 Midterm #1 – Solutions

1. (a) We find

$$\frac{1}{c} = \int_a^b \int_a^y (y-x)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \left[yx - \frac{x^2}{2} \right]_{x=a}^{x=y} \mathrm{d}y = \int_a^b \left(y^2 - \frac{y^2}{2} \right) - \left(ay - \frac{a^2}{2} \right) \mathrm{d}y$$

$$= \frac{1}{2} \int_a^b (y-a)^2\,\mathrm{d}y = \frac{1}{6}(y-a)^3 \Big|_a^b = \frac{1}{6}(b-a)^3$$

so that

$$c = \frac{6}{(b-a)^3}.$$

1. (b) By definition,

$$f_X(x) = \int_x^b c(y-x)\,\mathrm{d}y = c\left[\frac{1}{2}y^2 - xy\right]_{y=x}^{y=b} = c\left(\frac{b^2}{2} - bx - \frac{x^2}{2} + x^2\right) = \frac{c}{2}(x-b)^2 = \frac{3(x-b)^2}{(b-a)^3}$$

provided that $a \le x \le b$.

1. (c) By definition,

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{c(y-x)}{\frac{c}{2}(x-b)^2} = \frac{2(y-x)}{(x-b)^2}$$

provided that $x \le y \le b$.

2. (a) Let $Y = e^X$. For $y > 0$, the distribution function of Y is

$$F_Y(y) = P(Y \le y) = P(e^X \le y) = P(X \le \log y) = \int_{-\infty}^{\log y} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\} \mathrm{d}x$$

so that the density function of Y is

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{(\log y - \mu)^2}{2\sigma^2} \right\} \cdot \frac{1}{y} = \frac{1}{\sqrt{2\pi}\sigma y} \exp\left\{ -\frac{(\log y - \mu)^2}{2\sigma^2} \right\}$$

for $y > 0$. *We say that the random variable Y has a log-normal distribution with parameters μ and σ^2.*

2. (b) Let $Y = 1/X$. For $y > 0$, the distribution function of Y is

$$F_Y(y) = P(Y \le y) = P(1/X \le y) = P(X \ge 1/y) = 1 - P(X \le 1/y)$$
$$= 1 - \int_{-\infty}^{1/y} \frac{b^{-a}}{\Gamma(a)} x^{a-1} e^{-x/b}\,\mathrm{d}x$$

so that the density function of Y is

$$f_Y(y) = \frac{b^{-a}}{\Gamma(a)} y^{1-a} e^{-1/(by)} \cdot \frac{1}{y^2} = \frac{b^{-a}}{\Gamma(a)} y^{-a-1} e^{-1/(by)}$$

for $y > 0$. *We say that the random variable Y has an inverse gamma distribution with parameters a and $1/b$.*

3. If $U = \sqrt{XY}$ and $V = \sqrt{X/Y}$, then solving for X and Y gives $X = UV$ and $Y = U/V$. The Jacobian of this transformation is

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1/v & -u/v^2 \end{vmatrix} = -\frac{2u}{v}.$$

We now need to be careful with the limits of integration. Since $x > 1$ and $y > 1$ we see that necessarily $u > 1$ and $v > 0$. However, if $x = uv$, then $x > 1$ implies $v > 1/u$. If $y = u/v$, then $y > 1$ implies $u > v$. Thus, we conclude that $0 < 1/u < v < u$ and $u > 1$, and so for these values of u, v, we have

$$f_{U,V}(u,v) = f_{X,Y}(uv, u/v) \cdot |J| = \frac{9}{u^8} \cdot \frac{2u}{v} = \frac{18}{u^7 v}.$$

The marginal for U is given by

$$\begin{aligned} f_U(u) &= \int_{-\infty}^{\infty} f_{U,V}(u,v)\,\mathrm{d}v = \frac{18}{u^7}\int_{1/u}^{u} \frac{1}{v}\,\mathrm{d}v = \frac{18\log|v|}{u^7}\bigg|_{v=1/u}^{v=u} \\ &= \frac{18}{u^7}\left(\log|u| - \log|1/u|\right) = \frac{36\log|u|}{u^7} = \frac{36\log u}{u^7} \end{aligned}$$

provided $u > 1$.

4. By definition, we have

$$\begin{aligned} f_{X,Y}(x,y) = f_{Y|X=x}(y)f_X(x) &= \frac{1}{\sqrt{2\pi}}e^{-(y-x)^2/2} \cdot \frac{1}{\sqrt{2\pi}}e^{-x^2/2} = \frac{1}{2\pi}\exp\left\{-\frac{1}{2}(y^2 - 2xy + x^2 + x^2)\right\} \\ &= \frac{1}{2\pi}\exp\left\{-\frac{1}{2}(y^2 - 2xy + 2x^2)\right\} = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{y^2}{2}\right\} \cdot \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(2x^2 - 2xy)}{2}\right\} \\ &= \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{y^2}{2}\right\} \cdot \frac{1}{\sqrt{2\pi}}\exp\left\{-\left(x^2 - xy + \frac{y^2}{4} - \frac{y^2}{4}\right)\right\} \\ &= \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{y^2}{4}\right\} \cdot \frac{1}{\sqrt{2\pi}}\exp\left\{-\left(x - \frac{y}{2}\right)^2\right\} \\ &= \frac{\sqrt{0.5}}{\sqrt{2\pi}}\exp\left\{-\frac{y^2}{4}\right\} \cdot \frac{1}{\sqrt{2\pi}\sqrt{0.5}}\exp\left\{-\frac{1}{2\cdot 0.5}\left(x - \frac{y}{2}\right)^2\right\} \\ &= \frac{1}{\sqrt{2\pi}\sqrt{2}}\exp\left\{-\frac{y^2}{2\cdot 2}\right\} \cdot \frac{1}{\sqrt{2\pi}\sqrt{0.5}}\exp\left\{-\frac{1}{2\cdot 0.5}\left(x - \frac{y}{2}\right)^2\right\} \end{aligned}$$

so that

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}\sqrt{2}}\exp\left\{-\frac{y^2}{4}\right\}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sqrt{0.5}}\exp\left\{-\frac{1}{2\cdot 0.5}\left(x - \frac{y}{2}\right)^2\right\}\mathrm{d}x.$$

Notice that we have written this in such a way that the resulting integral equals 1. (It is the integral of the density function of a $N(y/2, 1/2)$ random variable.) Therefore,

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sqrt{2}}\exp\left\{-\frac{y^2}{2\cdot 2}\right\}$$

for $-\infty < y < \infty$ which verifies, in fact, that $Y \in N(0,2)$.