Statistics 351 Fall 2008 Midterm #1 – Solutions

1. (a) By definition,

$$f_X(x) = \int_x^\infty e^{-y}\,\mathrm{d}y = \left(-e^{-y}\right)\Big|_x^\infty = e^{-x}, \quad x > 0.$$

Note that $X \in \text{Exp}(1)$ so that $\mathbb{E}(X) = 1$.

1. (b) By definition,

$$f_Y(y) = \int_0^y e^{-y}\,\mathrm{d}x = ye^{-y}, \quad y > 0.$$

Note that $Y \in \Gamma(2, 1)$.

1. (c) By definition,

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{e^{-y}}{e^{-x}} = e^{x-y}, \quad 0 < x < y < \infty.$$

1. (d) By definition,

$$E(Y|X=x) = \int_{-\infty}^\infty y \cdot f_{Y|X=x}(y)\,\mathrm{d}y = \int_x^\infty y \cdot e^{x-y}\,\mathrm{d}y.$$

Let $u = y - x$ so that $\mathrm{d}u = \mathrm{d}y$ and the integral above becomes

$$\int_x^\infty y \cdot e^{x-y}\,\mathrm{d}y = \int_0^\infty (u+x)e^{-u}\,\mathrm{d}u = \int_0^\infty ue^{-u}\,\mathrm{d}u + x\int_0^\infty e^{-u}\,\mathrm{d}u = \Gamma(2) + x\Gamma(1) = 1 + x$$

and so $\mathbb{E}(Y|X) = 1 + X$.

1. (e) Using **(d)** we find $\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|X)) = \mathbb{E}(1+X) = 1 + \mathbb{E}(X)$. However, from **(a)** we know that $\mathbb{E}(X) = 1$ and so

$$\mathbb{E}(Y) = 1 + \mathbb{E}(X) = 1 + 1 = 2.$$

1. (f) If $a > 1$, then we find

$$\begin{aligned} P\{aX < Y\} = \iint_{\{ax<y\}} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^\infty \int_{ax}^\infty e^{-y}\,\mathrm{d}y\,\mathrm{d}x &= \int_0^\infty \left(-e^{-y}\right)\Big|_{ax}^\infty \mathrm{d}x \\ &= \int_0^\infty e^{-ax}\,\mathrm{d}x \\ &= \left(-\frac{1}{a}e^{-ax}\right)\Big|_0^\infty \\ &= \frac{1}{a}. \end{aligned}$$

1. (g) If $U = X + Y$ and $V = Y$, then solving for X and Y gives

$$X = U - V \quad \text{and} \quad Y = V.$$

The Jacobian of this transformation is

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1.$$

Therefore, we conclude

$$f_{U,V}(u,v) = f_{X,Y}(u-v,v) \cdot |J| = e^{-v} \cdot 1 = e^{-v}$$

provided that $0 < v < u < 2v < \infty$ (or, equivalently, $\frac{u}{2} < v < u$). The marginal for U is given by

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,\mathrm{d}v = \int_{u/2}^{u} e^{-v}\,\mathrm{d}v = \left(-e^{-v}\right)\Big|_{u/2}^{u} = e^{-u/2} - e^{-u}, \quad u > 0.$$

2. By the law of total probability,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X|Y=y}(x) f_Y(y)\,\mathrm{d}y = \int_0^{\infty} \frac{y^q}{\Gamma(q)} x^{q-1} e^{-yx} \cdot \frac{a^p}{\Gamma(p)} y^{p-1} e^{-ay}\,\mathrm{d}y$$
$$= \frac{a^p}{\Gamma(p)\Gamma(q)} x^{q-1} \int_0^{\infty} y^{p+q-1} e^{-y(x+a)}\,\mathrm{d}y.$$

We now recognize

$$\int_0^{\infty} y^{p+q-1} e^{-y(x+a)}\,\mathrm{d}y$$

as a gamma function. That is, make the change of variables $u = y(x+a)$ so that $\mathrm{d}u = (x+a)\,\mathrm{d}y$ and

$$\int_0^{\infty} y^{p+q-1} e^{-y(x+a)}\,\mathrm{d}y = \int_0^{\infty} \left(\frac{u}{x+a}\right)^{p+q-1} e^{-u} \frac{\mathrm{d}u}{x+a} = \left(\frac{1}{x+a}\right)^{p+q} \int_0^{\infty} u^{p+q-1} e^{-u}\,\mathrm{d}u$$
$$= \frac{\Gamma(p+q)}{(x+a)^{p+q}}.$$

Hence, we conclude that

$$f_X(x) = \frac{a^p}{\Gamma(p)\Gamma(q)} x^{q-1} \cdot \frac{\Gamma(p+q)}{(x+a)^{p+q}} = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \frac{a^p x^{q-1}}{(x+a)^{p+q}}, \quad 0 < x < \infty.$$

This is sometimes called a type II generalized Pareto density with parameters q, p, a.

3. (a) Notice that

$$X_{n+1} = \left(\frac{1-p}{p}\right)^{S_{n+1}} = \left(\frac{1-p}{p}\right)^{S_n+Y_{n+1}} = \left(\frac{1-p}{p}\right)^{S_n}\left(\frac{1-p}{p}\right)^{Y_{n+1}}$$

Therefore,

$$\begin{aligned}\mathbb{E}(X_{n+1}|X_0,\ldots,X_n) &= \mathbb{E}\left(\left(\frac{1-p}{p}\right)^{S_n}\left(\frac{1-p}{p}\right)^{Y_{n+1}}|X_0,\ldots,X_n\right)\\ &= \left(\frac{1-p}{p}\right)^{S_n}\mathbb{E}\left(\left(\frac{1-p}{p}\right)^{Y_{n+1}}|X_0,\ldots,X_n\right)\\ &= \left(\frac{1-p}{p}\right)^{S_n}\mathbb{E}\left(\left(\frac{1-p}{p}\right)^{Y_{n+1}}\right)\end{aligned}$$

where the second equality follows from "taking out what is known" and the third equality follows from the fact that $Y_1, Y_2, \ldots$ are independent. We now compute

$$\mathbb{E}\left(\left(\frac{1-p}{p}\right)^{Y_{n+1}}\right) = p\left(\frac{1-p}{p}\right)^{1} + (1-p)\left(\frac{1-p}{p}\right)^{-1} = (1-p)+p = 1$$

and so we conclude

$$\mathbb{E}(X_{n+1}|X_0,\ldots,X_n) = \left(\frac{1-p}{p}\right)^{S_n} = X_n.$$

Hence, $\{X_n, n = 0,1,2,\ldots\}$ is, in fact, a martingale

3. (b) Since X_n is a martingale, we know that $\mathbb{E}(X_{n+1}|X_n) = X_n$. Therefore, using properties of conditional expectation we find

$$\mathbb{E}(X_{n+1}) = \mathbb{E}(\mathbb{E}(X_{n+1}|X_n)) = \mathbb{E}(X_n) = \cdots = \mathbb{E}(X_0).$$

Since $S_0 = 0$ we see that $X_0 = 1$ so that $\mathbb{E}(X_0) = 1$ and therefore $E(X_n) = 1$ for all $n = 0,1,2,\ldots$.

4. If $U = Y - X$ and $V = X$, then solving for X and Y gives

$$X = V \quad \text{and} \quad Y = U + V.$$

The Jacobian of this transformation is

$$J = \begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} = \begin{vmatrix}0 & 1\\ 1 & 1\end{vmatrix} = -1.$$

Therefore, we conclude

$$f_{U,V}(u,v) = f_{X,Y}(v,u+v)\cdot|J| = c(c+1)(b-a)^c u^{-c-2}\cdot 1 = c(c+1)(b-a)^c u^{-c-2}$$

provided that $-\infty < v < a$ and $-\infty < b - v < u < \infty$.

You should check that $\int_{-\infty}^{a}\int_{b-v}^{\infty} c(c+1)(b-a)^c u^{-c-2}\,\mathrm{d}u\,\mathrm{d}v = 1.$