# Statistics 351 Midterm #1 – October 6, 2008

**This exam has 4 problems and 6 numbered pages.**

*You have 50 minutes to complete this exam. Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit.* ***Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.*** *Points will be deducted for incoherent, incorrect, and/or irrelevant statements.*

*This exam is closed-book, except that one* $8\frac{1}{2} \times 11$ *double-sided page of handwritten notes is permitted. No other aids are allowed.*

*You must answer all of the questions in the space provided. Note that blank space is NOT an indication of a question's difficulty.*

Name: ____________________________________

Instructor: Michael Kozdron

| Problem | Score |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |

TOTAL: ____________

**1.** (*24 points*) Suppose that the random vector $(X, Y)'$ has joint density function

$$f_{X,Y}(x,y) = \begin{cases} e^{-x}, & \text{if } 0 < y < x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

**(a)** Determine $f_Y(y)$, the marginal density function of $Y$, and $\mathbb{E}(Y)$, the expected value of $Y$.

**(b)** Determine $f_X(x)$, the marginal density function of $X$.

**(c)** Calculate $f_{Y|X=x}(y)$, the conditional density function of $Y$ given $X = x$.

*(continued)*

Recall that the joint density function of $(X,Y)'$ is $f_{X,Y}(x,y) = \begin{cases} e^{-x}, & \text{if } 0 < y < x < \infty, \\ 0, & \text{otherwise.} \end{cases}$

**(d)** Use the result of **(c)** to determine $\mathbb{E}(Y|X)$.

**(e)** Use the results of **(a)** and **(d)** to calculate $\mathbb{E}(X)$.

**(f)** For $a < 1$, determine $P\{Y < aX\}$.

*(continued)*

Recall that the joint density function of $(X,Y)'$ is $f_{X,Y}(x,y) = \begin{cases} e^{-x}, & \text{if } 0 < y < x < \infty, \\ 0, & \text{otherwise.} \end{cases}$

**(g)** Determine the density function of the random variable $X+Y$.
*Hint: Let* $U = X+Y,\ V = 2Y$.

**2.** (*10 points*) Suppose that $Y_1, Y_2, \ldots$ are independent and identically distributed random variables with $P\{Y_1 = 1\} = p$, $P\{Y_1 = -1\} = 1 - p$ for some $0 < p < 1/2$. Let $S_0 = 0$ and for $n = 1, 2, \ldots$, let $S_n = Y_1 + \cdots + Y_n$ denote their partial sums. Finally, for $n = 0, 1, 2, \ldots$, define

$$X_n = \left(\frac{1-p}{p}\right)^{S_n}.$$

**(a)** Carefully verify that $\{X_n, n = 0, 1, 2, \ldots\}$ is a martingale.

**(b)** Compute $\mathbb{E}(X_n)$, $n = 0, 1, 2, \ldots$.

**3.** (*6 points*) The random vector $(X, Y)'$ is said to have a bilateral bivariate Pareto distribution with parameters $a$, $b$, $c$, written $(X, Y) \in \text{Pabb}(a, b, c)$, if the joint density function is given by

$$f_{X,Y}(x, y) = c(c+1)(b-a)^c(y-x)^{-c-2}$$

provided that $c > 0$ and $-\infty < x < a < b < y < \infty$. If $U = Y - X$ and $V = X$, determine $f_{U,V}(u, v)$, the joint density function of $(U, V)'$.

**4.** (*10 points*) Suppose that the random variable $Y \in \Gamma(p, 1/a)$ where $p > 0$ and $a > 0$ are parameters so that the density function of $Y$ is

$$f_Y(y) = \frac{a^p}{\Gamma(p)} y^{p-1} e^{-ay}, \quad y > 0.$$

Suppose further that $X|Y = y \in \Gamma(q, 1/y)$ where $q > 0$ is a parameter so that

$$f_{X|Y=y}(x) = \frac{y^q}{\Gamma(q)} x^{q-1} e^{-yx}, \quad x > 0.$$

Determine $f_X(x)$, the density function of $X$.