Lecture #17: The Joint Distribution of $X_{(1)}$ and $X_{(n)}$

Recall. Suppose that $X_1, \ldots, X_n$ are random variables. We define

$$X_{(1)} = \min\{X_1, \ldots, X_n\} \quad \text{and} \quad X_{(n)} = \max\{X_1, \ldots, X_n\}.$$

In general, $X_{(k)}$, $k = 1, \ldots, n$, denotes the kth smallest of $X_1, \ldots, X_n$. We call

$$(X_{(1)}, X_{(2)}, \ldots, X_{(n)})'$$

the *order statistic* and $X_{(k)}$ the *kth order variable*. Note that some texts call $X_{(k)}$ the *kth order statistic*.

If $X_1, \ldots, X_n$ are i.i.d. with common distribution function $F(x)$ and common denisty function $f(x)$, then

- $F_{X_{(1)}}(y_1) = 1 - [1 - F(y_1)]^n$ and $f_{X_{(1)}}(y_1) = nf(y_1)[1 - F(y_1)]^{n-1}$,
- $F_{X_{(n)}}(y_n) = [F(y_n)]^n$ and $f_{X_{(n)}}(y_n) = nf(y_n)[F(y_n)]^{n-1}$.

Note that even though $X_1, \ldots, X_n$ are independent, there is no reason to suspect that $X_{(1)}$ and $X_{(n)}$ are independent.

Consider $(X_{(1)}, X_{(n)})'$. We want to compute

$$F_{X_{(1)},X_{(n)}}(y_1, y_n) = P\{X_{(1)} \le y_1,\ X_{(n)} \le y_n\}.$$

To do so, we proceed in a manner similar to the one-dimensional case. Notice that

$$\begin{aligned} P\{X_{(1)} > y_1, X_{(n)} \le y_n\} &= P\{y_1 < X_k \le y_n \ \ \forall\ \ k = 1, \ldots, n\} \\ &= \prod_{k=1}^{n} P\{y_1 < X_k \le y_n\} \quad \text{since } X_1, \ldots, X_n \text{ are independent} \\ &= [P\{y_1 < X_1 \le y_n\}]^n \quad \text{since } X_1, \ldots, X_n \text{ are indentically distributed} \\ &= [F(y_n) - F(y_1)]^n \quad \text{provided } y_1 < y_n \end{aligned}$$

On the other hand, if $y_1 \ge y_n$, then $P\{X_{(1)} > y_1,\ X_{(n)} \le y_n\} = 0$.

We now observe that

$$P\{X_{(n)} \le y_n\} = P\{X_{(1)} \le y_1,\ X_{(n)} \le y_n\} + P\{X_{(1)} > y_1,\ X_{(n)} \le y_n\}$$

and so

$$F_{X_{(1)},X_{(n)}}(y_1, y_n) = P\{X_{(1)} \le y_1,\ X_{(n)} \le y_n\} = P\{X_{(n)} \le y_n\} - P\{X_{(1)} > y_1,\ X_{(n)} \le y_n\}.$$

Therefore, if $y_1 < y_n$, then

$$F_{X_{(1)},X_{(n)}}(y_1, y_n) = F_{X_{(n)}}(y_n) - [F(y_n) - F(y_1)]^n$$

and if $y_1 \ge y_n$, then

$$F_{X_{(1)},X_{(n)}}(y_1, y_n) = F_{X_{(n)}}(y_n).$$

Since $F_{X_{(n)}}(y_n) = [F(y_n)]^n$, we conclude the following.

Theorem. *If $X_1, \ldots, X_n$ are i.i.d. with common distribution function $F(x)$ and common denisty function $f(x)$, then the joint distribution function of $(X_{(1)}, X_{(n)})'$ is*

$$F_{X_{(1)},X_{(n)}}(y_1, y_n) = \begin{cases} [F(y_n)]^n - [F(y_n) - F(y_1)]^n, & \text{if } y_1 < y_n, \\ [F(y_n)]^n, & \text{if } y_1 \geq y_n. \end{cases}$$

Furthermore, if $X_1, \ldots, X_n$ are continuous random variables, then

$$\begin{aligned} f_{X_{(1)},X_{(n)}}(y_1, y_n) &= \frac{\partial^2}{\partial y_1 \partial y_n} F_{X_{(1)},X_{(n)}}(y_1, y_n) \\ &= \begin{cases} n(n-1)f(y_1)f(y_n)[F(y_n) - F(y_1)]^{n-2}, & \text{if } y_1 < y_n, \\ 0, & \text{if } y_1 \geq y_n. \end{cases} \end{aligned}$$

Distribution of the Range

Let $R = X_{(n)} - X_{(1)}$ denote the *range* which gives some information about how spread out the distribution F is. We can determine the distribution of R by combining our earlier results with the techniques of Chapter 1.

Let $R = X_{(n)} - X_{(1)}$ and $V = X_{(1)}$ so that

$$X_{(1)} = V \quad \text{and} \quad X_{(n)} = R + V.$$

The Jacobian of this transformation is

$$J = \begin{vmatrix} \dfrac{\partial y_1}{\partial r} & \dfrac{\partial y_1}{\partial v} \\ \dfrac{\partial y_n}{\partial r} & \dfrac{\partial y_n}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1.$$

Therefore,

$$f_{R,V}(r, v) = f_{X_{(1)},X_{(n)}}(v, r + v) \cdot 1 = n(n-1)f(v)f(r+v)[F(r+v) - F(v)]^{n-2}$$

provided $v < v + r$, i.e., $r > 0$. Integrating implies that

$$f_R(r) = \int_{-\infty}^{\infty} f_{V,R}(v, r)\,\mathrm{d}v = n(n-1)\int_{-\infty}^{\infty} f(v)f(r+v)[F(r+v) - F(v)]^{n-2}\,\mathrm{d}v, \quad r > 0.$$

Example. Suppose that $X_1, \ldots, X_n$ are i.i.d. Exp(1) random variables so that

$$F(x) = \begin{cases} 1 - e^{-x}, & \text{if } x > 0, \\ 0, & \text{if } x \geq 0, \end{cases} \quad \text{and} \quad f(x) = e^{-x}, \quad x > 0.$$

We compute

- $F_{X_{(1)}}(y_1) = 1 - [1 - (1 - e^{-y_1})]^n = 1 - e^{-ny_1}, \quad y_1 > 0,$

- $f_{X_{(1)}}(y_1) = ne^{-ny_1}, \quad y_1 > 0,$
- $F_{X_{(n)}}(y_n) = [1 - e^{-y_n}]^n, \quad y_n > 0,$
- $f_{X_{(n)}}(y_n) = ne^{-y_n}[1 - e^{-y_n}]^{n-1}, \quad y_n > 0,$
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$$F_{X_{(1)},X_{(n)}}(y_1, y_n) = \begin{cases} [1 - e^{-y_n}]^n - [e^{-y_1} - e^{-y_n}]^n, & \text{if } 0 < y_1 < y_n < \infty, \\ [1 - e^{-y_n}]^n, & \text{otherwise,} \end{cases}$$
- $f_{X_{(1)},X_{(n)}}(y_1, y_n) = n(n-1)e^{-y_1}e^{-y_n}[e^{-y_1} - e^{-y_n}]^{n-2}, \quad 0 < y_1 < y_n < \infty,$
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$$\begin{aligned} f_R(r) &= n(n-1)\int_0^\infty [e^{-v} - e^{-(r+v)}]^{n-2} e^{-v-(r+v)}\,\mathrm{d}v \\ &= n(n-1)e^{-r}[1 - e^{-r}]^{n-2}\int_0^\infty e^{-v(n-2)}e^{-2v}\,\mathrm{d}v \\ &= n(n-1)e^{-r}[1 - e^{-r}]^{n-2}\int_0^\infty e^{-vn}\,\mathrm{d}v \\ &= n(n-1)e^{-r}[1 - e^{-r}]^{n-2} \cdot \frac{1}{n}e^{-vn}\Big|_0^\infty \\ &= (n-1)e^{-r}[1 - e^{-r}]^{n-2}, \quad r > 0. \end{aligned}$$