Lecture #11: Distributions with Random Parameters

Example. Suppose that the random vector $(X, Y)'$ has joint density function

$$f_{X,Y}(x,y) = \begin{cases} e^{-y}, & \text{if } 0 < x < y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine $f_X(x)$, the marginal density function of X.

(b) Determine $f_Y(y)$, the marginal density function of Y.

(c) Calculate $f_{X|Y=y}(x)$, the conditional density function of X given $Y = y$.

(d) Calculate $f_{Y|X=x}(y)$, the conditional density function of Y given $X = x$.

Solution. For **(a)** we have, by definition, that

$$f_X(x) = \int_x^\infty e^{-y}\,\mathrm{d}y = \left(-e^{-y}\right)\Big|_x^\infty = e^{-x}, \quad x > 0,$$

implying that $X \in \text{Exp}(1)$. For **(b)** we have, by definition, that

$$f_Y(y) = \int_0^y e^{-y}\,\mathrm{d}x = ye^{-y}, \quad y > 0,$$

implying that $Y \in \Gamma(2, 1)$. Thus, for **(c)** we conclude that

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{e^{-y}}{ye^{-y}} = \frac{1}{y}, \quad 0 < x < y,$$

implying that $X|Y = y \in U(0, y)$. Finally, for **(d)** we find

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{e^{-y}}{e^{-x}} = e^{x-y}, \quad 0 < x < y < \infty.$$

Example. Suppose that $(X, Y)'$ is a jointly distributed random variable with density function

$$f_{X,Y}(x,y) = \begin{cases} cxy, & \text{if } 0 < y < 1 \text{ and } 0 < x < y^2 < 1, \\ 0, & \text{otherwise,} \end{cases}$$

where the value of the normalizing constant c is chosen so that $\int_{-\infty}^\infty \int_{-\infty}^\infty f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y = 1$.

(a) Determine the value of c.

(b) Compute $f_X(x)$.

(c) Compute $f_Y(y)$.

(d) Compute $f_{Y|X=x}(y)$.

Solution. **(a)** We find

$$\int_0^1 \int_{\sqrt{x}}^1 xy \,\mathrm{d}y\,\mathrm{d}x = \int_0^1 x \cdot \frac{1}{2}y^2 \Big|_{y=\sqrt{x}}^{y=1} \mathrm{d}x = \frac{1}{2}\int_0^1 x(1-x)\,\mathrm{d}x = \frac{1}{2}\left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1 = \frac{1}{12}$$

so that $c = 12$.

(b) By definition,

$$f_X(x) = \int_{\sqrt{x}}^1 12xy\,\mathrm{d}y = 12x \cdot \frac{1}{2}y^2\Big|_{y=\sqrt{x}}^{y=1} = 6x(1-x)$$

provided that $0 < x < 1$.

(c) By definition,

$$f_Y(y) = \int_0^{y^2} 12xy\,\mathrm{d}y = 12y \cdot \frac{1}{2}x^2\Big|_{x=0}^{x=y^2} = 6y^5$$

provided that $0 < y < 1$.

(d) By definition, if $0 < x < 1$ is fixed, then

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{12xy}{6x(1-x)} = \frac{2y}{1-x}$$

provided that $\sqrt{x} < y < 1$.

Last class we introduced the *law of total probability*. It turns out that if X and Y are jointly distributed random variables, then we can generalize this result.

Suppose that X is continuous.

- If Y is also continuous, then

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y = \int_{-\infty}^{\infty} f_{X|Y=y}(x) f_Y(y)\,\mathrm{d}y.$$

- However, if Y is discrete, then

$$f_X(x) = \sum_y f_{X,Y}(x,y) = \sum_y f_{X|Y=y}(x) P\{Y = y\}.$$

On the other hand, suppose that X is discrete.

- If Y is continuous, then

$$P\{X = x\} = \int_{-\infty}^{\infty} P\{X = x|Y = y\} f_Y(y)\,\mathrm{d}y.$$

- However, if Y is also discrete, then

$$P\{X = x\} = \sum_y P\{X = x|Y = y\} P\{Y = y\}.$$

Example. Suppose that $X|M = m \in \mathrm{Po}(m)$ with $M \in \mathrm{Exp}(1)$. Determine the (unconditional) distribution of X.

Solution. By the law of total probability, we find that for $k = 0, 1, 2, \ldots,$

$$P\{X = k\} = \int_0^{\infty} P\{X = k|M = x\} f_M(x)\,\mathrm{d}x = \int_0^{\infty} \frac{e^{-x}x^k}{k!} \cdot e^{-x}\,\mathrm{d}x = \frac{1}{k!}\int_0^{\infty} x^k e^{-2x}\,\mathrm{d}x.$$

Making the substitution $u = 2x,\ \mathrm{d}u = 2\,\mathrm{d}x$ gives

$$\frac{1}{k!}\int_0^{\infty} x^k e^{-2x}\,\mathrm{d}x = \frac{1}{k!}\int_0^{\infty} u^k 2^{-k} e^{-u} 2^{-1}\,\mathrm{d}u = \frac{1}{2^{k+1}k!}\int_0^{\infty} u^k e^{-u}\,\mathrm{d}u = \frac{\Gamma(k+1)}{2^{k+1}k!} = \frac{1}{2^{k+1}}$$

since $\Gamma(k+1) = k!$. Hence, we see that $P\{X = k\} = 2^{-k-1}$, $k = 0, 1, 2, \ldots,$ and so we conclude that $X \in \mathrm{Ge}(1/2)$.