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Lecture #9: The Gamma Function

Suppose that $p > 0$, and define

$$\Gamma(p) := \int_0^\infty u^{p-1}\, e^{-u}\, du.$$

We call $\Gamma(p)$ the *Gamma function* and it appears in many of the formulæ of density functions for continuous random variables such as the Gamma distribution, Beta distribution, Chi-squared distribution, t distribution, and F distribution.

The first thing that should be checked is that the integral defining $\Gamma(p)$ is convergent for $p > 0$. For now, we will assume that it is true that the Gamma function is well-defined. This will allow us to derive some of its important properties and show its utility for statistics.

The Gamma function may be viewed as a generalization of the factorial function as this first result shows.

Proposition 1. *If $p > 0$, then $\Gamma(p+1) = p\,\Gamma(p)$.*

Proof. This is proved using integration by parts from first-year calculus. Indeed,

$$\Gamma(p+1) = \int_0^\infty u^{p+1-1}\, e^{-u}\, du = \int_0^\infty u^p\, e^{-u}\, du = -u^p e^{-u}\Big|_0^\infty + \int_0^\infty p u^{p-1}\, e^{-u}\, du = 0 + p\,\Gamma(p).$$

To do the integration by parts, let $w = u^p$, $dw = pu^{p-1}$, $dv = e^{-u}$, $v = -e^{-u}$ and recall that $\int w\, dv = wv - \int v\, dw$. □

If p is an integer, then we have the following corollary.

Corollary 2. *If n is a positive integer, then $\Gamma(n) = (n-1)!$.*

Proof. Using the previous proposition, we see that

$$\Gamma(n) = (n-1)\,\Gamma(n-1) = (n-1)(n-2)\,\Gamma(n-2) = \cdots = (n-1)(n-2)\cdots 2\cdot\Gamma(1).$$

However,

$$\Gamma(1) = \int_0^\infty u^0 e^{-u}\, du = \int_0^\infty e^{-u}\, du = -e^{-u}\Big|_0^\infty = 1 \tag{1}$$

and so

$$\Gamma(n) = (n-1)(n-2)\cdots 2\cdot 1 = (n-1)!$$

as required. □

The next proposition shows us how to calculate $\Gamma(p)$ for certain fractional values of p.

Proposition 3. $\Gamma(1/2) = \sqrt{\pi}$.

Proof. By definition,
$$\Gamma(1/2) = \int_0^\infty u^{-1/2} e^{-u}\, du.$$
Making the substitution $u = v^2$ so that $du = 2v\, dv$ gives
$$\int_0^\infty u^{-1/2} e^{-u}\, du = \int_0^\infty v^{-1} e^{-v^2}\, 2v\, dv = 2\int_0^\infty e^{-v^2}\, dv = \int_{-\infty}^\infty e^{-v^2}\, dv$$
where the last equality follows since e^{-v^2} is an even function. We now recognize this as the density function of a $\mathcal{N}(0, 1/2)$ random variable. That is,
$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{v^2}{2\sigma^2}}\, dv = 1$$
and so
$$\int_{-\infty}^\infty e^{-\frac{v^2}{2\sigma^2}}\, dv = \sigma\sqrt{2\pi}.$$
Choosing $\sigma^2 = 1/2$ gives
$$\int_{-\infty}^\infty e^{-v^2}\, dv = \sqrt{\pi}$$
and so we conclude that $\Gamma(1/2) = \sqrt{\pi}$ as claimed. □

This proposition can be combined with Proposition 1 to show, for example, that
$$\Gamma(3/2) = \Gamma(1/2 + 1) = 1/2 \cdot \Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$
and
$$\Gamma(5/2) = \Gamma(3/2 + 1) = 3/2 \cdot \Gamma(3/2) = \frac{3\sqrt{\pi}}{4}.$$

For students, though, perhaps the most powerful use of the Gamma function is to compute integrals such as the following.

Example 4. Suppose that $Y \sim \text{Exp}(\theta)$. Use Gamma functions to quickly compute $\mathbb{E}(Y^2)$.

Solution. By definition, we have
$$\mathbb{E}(Y^2) = \int_{-\infty}^\infty y^2 f_Y(y)\, dy = \frac{1}{\theta}\int_0^\infty y^2 e^{-y/\theta}\, dy.$$
Make the substitution $u = y/\theta$ so that $dy = \theta du$. This gives
$$\frac{1}{\theta}\int_0^\infty y^2 e^{-y/\theta}\, dy = \frac{1}{\theta}\int_0^\infty \theta^2 u^2 e^{-u}\, \theta\, du = \theta^2 \int_0^\infty u^2 e^{-u}\, du = \theta^2\, \Gamma(3).$$
By Corollary 2, $\Gamma(3) = (3-1)! = 2$ and so $\mathbb{E}(Y^2) = 2\,\theta^2$.

Example 5. If $Y \sim \text{Exp}(\theta)$, then this method can be applied to compute $\mathbb{E}(Y^k)$ for any positive integer k. Indeed,

$$\mathbb{E}(Y^k) = \frac{1}{\theta}\int_0^\infty y^k e^{-y/\theta}\,dy = \frac{1}{\theta}\int_0^\infty \theta^k u^k e^{-u}\,\theta\,du = \theta^k\,\Gamma(k+1) = k!\,\theta^k.$$

Example 6. If $X \in \mathcal{N}(0,1)$, determine the distribution of X^2.

Solution. Let $Y = g(X) = X^2$ so that

$$F_Y(y) = P\{Y \le y\} = P\{X^2 \le y\}.$$

Notice that $g(x) = x^2$ is NOT monotone on $-\infty < x < \infty$. Therefore,

$$\begin{aligned} P\{X^2 \le y\} = P\{-\sqrt{y} \le X \le \sqrt{y}\} &= \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,\mathrm{d}x \\ &= \int_{-\sqrt{y}}^{0} f_X(x)\,\mathrm{d}x + \int_0^{\sqrt{y}} f_X(x)\,\mathrm{d}x \\ &= \int_0^{\sqrt{y}} f_X(x)\,\mathrm{d}x - \int_0^{-\sqrt{y}} f_X(x)\,\mathrm{d}x, \end{aligned}$$

and so

$$f_Y(y) = F_Y'(y) = f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f_X(-\sqrt{y}) \cdot \frac{-1}{2\sqrt{y}} = \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) - f_X(-\sqrt{y})\right].$$

Since $g(x) = x^2$ is two-to-one we have two terms. In particular,

$$f_Y(y) = \frac{1}{\sqrt{2\pi}}e^{-y/2}\cdot\frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}}e^{-y/2}\cdot\frac{1}{2\sqrt{y}} = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{y}}e^{-y/2} = \frac{1}{2^{1/2}\Gamma(1/2)}y^{1/2-1}e^{-y/2},\ \ y > 0,$$

i.e., $Y \in \Gamma(1/2, 2)$ or $Y \in \chi^2(1)$.

Remark. We will not cover many-to-one functions beyond this example. Therefore, omit everything following Example 2.6 on page 23 (including Example 2.7 on page 24).

Theorem 7. *For $p > 0$, the integral*

$$\int_0^\infty u^{p-1}\,e^{-u}\,du$$

is absolutely convergent.

Proof. Since we are considering the value of the improper integral

$$\int_0^\infty u^{p-1}\,e^{-u}\,du$$

for all $p > 0$, there is need to be careful at both endpoints 0 and ∞.

We begin with the easiest case. If $p = 1$, then

$$\int_0^\infty u^0 e^{-u}\, du = \int_0^\infty e^{-u}\, du = \lim_{N\to\infty} \int_0^N e^{-u}\, du = \lim_{N\to\infty} (1 - e^{-N}) = 1.$$

For the remaining cases $0 < p < 1$ and $p > 1$ we will consider the integral from 0 to 1 and the integral from 1 to ∞ separately.

If $0 < p < 1$, then the integral

$$\int_0^1 u^{p-1}\, e^{-u}\, du$$

is improper. Thus,

$$\int_0^1 u^{p-1}\, e^{-u}\, du = \lim_{a\to 0+} \int_a^1 u^{p-1}\, e^{-u}\, du \le \lim_{a\to 0+} \int_a^1 u^{p-1}\, du = \lim_{a\to 0+} \frac{1 - a^p}{p} = \frac{1}{p}$$

since $e^{-u} \le 1$ for $0 \le u \le 1$.

Furthermore, if $0 < p < 1$, then $0 < u^{p-1} \le 1$ for $u \ge 1$ and so

$$\int_1^\infty u^{p-1}\, e^{-u}\, du = \lim_{N\to\infty} \int_1^N u^{p-1}\, e^{-u}\, du \le \lim_{N\to\infty} \int_1^N e^{-u}\, du = \lim_{N\to\infty} (1 - e^{-N}) = 1.$$

Thus, we can conclude that for $0 < p < 1$,

$$\int_0^\infty u^{p-1}\, e^{-u}\, du = \int_0^1 u^{p-1}\, e^{-u}\, du + \int_1^\infty u^{p-1}\, e^{-u}\, du \le \frac{1}{p} + 1 < \infty.$$

If $p > 1$, then $u^{p-1} \in [0, 1]$ and $e^{-u} \le 1$ for $0 \le u \le 1$. Thus,

$$\int_0^1 u^{p-1}\, e^{-u}\, du \le \int_0^1 u^{p-1}\, du = \frac{u^p}{p}\bigg|_0^1 = \frac{1}{p}.$$

On the other hand, if $p > 1$, then notice that $p - \lfloor p \rfloor \in [0, 1)$ so that $0 < u^{p-\lfloor p\rfloor -1} \le 1$ for $u \ge 1$. We then have

$$\int_1^N u^{p-1}\, e^{-u}\, du = \int_1^N u^{p-\lfloor p\rfloor-1} u^{\lfloor p\rfloor}\, e^{-u}\, du \le \int_1^N u^{\lfloor p\rfloor}\, e^{-u}\, du.$$

Thus, integration by parts $\lfloor p \rfloor$ times (the so-called *reduction formula*) gives

$$\begin{aligned}
&\int_1^N u^{\lfloor p\rfloor}\, e^{-u}\, du \\
&\qquad = -e^{-u}\left(u^{\lfloor p\rfloor} + \lfloor p\rfloor u^{\lfloor p\rfloor-1} + \lfloor p\rfloor\cdot(\lfloor p\rfloor - 1)u^{\lfloor p\rfloor-2} + \cdots + \lfloor p\rfloor\cdot(\lfloor p\rfloor-1)\cdots 2\cdot u\right)\bigg|_1^N \\
&\qquad\quad + \lfloor p\rfloor\cdot(\lfloor p\rfloor-1)\cdots 2\cdot 1\cdot \int_1^N e^{-u}\, du
\end{aligned}$$

and so

$$\lim_{N \to \infty} \int_1^N u^{\lfloor p \rfloor} \, e^{-u} \, du = \lfloor p \rfloor \, !.$$

Thus, we can conclude that for $p > 1$,

$$\int_0^\infty u^{p-1} \, e^{-u} \, du = \int_0^1 u^{p-1} \, e^{-u} \, du + \int_1^\infty u^{p-1} \, e^{-u} \, du \leq \frac{1}{p} + \lfloor p \rfloor \, ! < \infty.$$

In every case we have $u^{p-1} \, e^{-u} \geq 0$ and so

$$\int_0^\infty \left| u^{p-1} \, e^{-u} \right| \, du = \int_0^\infty u^{p-1} \, e^{-u} \, du < \infty.$$

That is, this integral is absolutely convergent, and so $\Gamma(p)$ is well-defined for $p > 0$. $\square$