Lecture #8: Functions of Multivariate Random Variables

Example (Chapter 1, Problem #31)**.** Suppose that $(X, Y)'$ have joint density

$$f(x,y) = \begin{cases} xe^{-x-xy} & \text{for } x > 0,\ y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the distribution of $X(1+Y)$.

Solution. Let $U = X(1+Y)$ and $V = 1+Y$ so that solving for X and Y gives

$$X = \frac{U}{V} \quad \text{and} \quad Y = V - 1.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v^{-1} & -uv^{-2} \\ 0 & 1 \end{vmatrix} = v^{-1}.$$

The density of $(U, V)'$ is therefore given by

$$f_{U,V}(u,v) = f_{X,Y}(u/v, v-1) \cdot |J| = \frac{u}{v} e^{-u} \cdot \frac{1}{|v|} = \frac{u}{v^2} e^{-u}$$

for $u > 0$ and $v > 1$. The marginal for U is

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\, \mathrm{d}v = \int_1^{\infty} \frac{u}{v^2} e^{-u}\, \mathrm{d}v = ue^{-u} \int_1^{\infty} \frac{1}{v^2}\, \mathrm{d}v = ue^{-u}$$

for $u > 0$ (which happens to be the density function of a $\Gamma(2,1)$ random variable). As an added bonus, we have also shown that $X(1+Y)$ and $(1+Y)$ are independent.

Example. Suppose that the random vector $(X, Y)'$ has joint density function

$$f_{X,Y}(x,y) = \begin{cases} e^{-y}, & \text{if } 0 < x < y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the density function of the random variable $X + Y$.

Solution. If $U = X + Y$ and $V = Y$, then solving for X and Y gives

$$X = U - V \quad \text{and} \quad Y = V.$$

The Jacobian of this transformation is

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1.$$

Therefore, we conclude

$$f_{U,V}(u,v) = f_{X,Y}(u-v, v) \cdot |J| = e^{-v} \cdot 1 = e^{-v} \qquad (*)$$

provided that $0 < v < u < 2v < \infty$ (or, equivalently, $\frac{u}{2} < v < u$). The marginal for U is given by

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,\mathrm{d}v = \int_{u/2}^{u} e^{-v}\,\mathrm{d}v = \left(-e^{-v}\right)\Big|_{u/2}^{u} = e^{-u/2} - e^{-u}, \quad u > 0.$$

In particular, note that U and V are NOT independent, even though $(*)$ suggests that they are. The reason, of course, is because of the dependence of u on v in the limits of integration.

Example. The purpose of this example is to prove a relationship between the gamma function and the beta integral that was used last lecture. That is, show that if $a > 0$ and $b > 0$, then

$$\int_0^1 x^{a-1}(1-x)^{b-1}\,\mathrm{d}x = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Proof. The solution is to consider the product $\Gamma(a)\Gamma(b)$ as a double integral, change variables, and simplify. That is,

$$\Gamma(a)\Gamma(b) = \left[\int_0^{\infty} x^{a-1}e^{-x}\,\mathrm{d}x\right]\left[\int_0^{\infty} y^{b-1}e^{-y}\,\mathrm{d}y\right] = \int_0^{\infty}\int_0^{\infty} x^{a-1}y^{b-1}e^{-(x+y)}\,\mathrm{d}x\,\mathrm{d}y.$$

Let $u = x+y$ and $v = \frac{x}{x+y}$ so that $x = uv$ and $y = u(1-v)$. The Jacobian of this transformation is

$$J = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -u$$

implying

$$\begin{aligned} \int_0^{\infty}\int_0^{\infty} x^{a-1}y^{b-1}e^{-(x+y)}\,\mathrm{d}x\,\mathrm{d}y &= \int_0^{\infty}\int_0^{1} (uv)^{a-1}(u(1-v))^{b-1}e^{-u}u\,\mathrm{d}u\,\mathrm{d}v \\ &= \left[\int_0^{\infty} u^{a+b-1}e^{-u}\,\mathrm{d}u\right]\left[\int_0^{1} v^{a-1}(1-v)^{b-1}\,\mathrm{d}v\right] \\ &= \Gamma(a+b)\int_0^1 v^{a-1}(1-v)^{b-1}\,\mathrm{d}v \end{aligned}$$

as required. □