Lecture #7: Functions of Multivariate Random Variables

Example (Chapter 1, Problem #8)**.** Suppose that $X \in \mathcal{N}(0,1)$ and $Y \in \mathcal{N}(0,1)$ are independent random variables. Show that $X/Y \in C(0,1)$.

Solution. We start with a bivariate random vector $(X,Y)'$ (i.e., two random variables), but we want the distribution of just one random variable, namely X/Y.

The "trick" is to let $U = X/Y$ and to introduce an auxiliary variable V which may be arbitrarily chosen. (Although it may be arbitrary, choose it suitably!)

Let $U = X/Y$ and $V = Y$ so that solving for X and Y gives

$$X = UV \quad \text{and} \quad Y = V.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v.$$

The density of $(U,V)'$ is therefore given by

$$f_{U,V}(u,v) = f_{X,Y}(uv,v) \cdot |J| = |v| f_X(uv) f_Y(v)$$

using the assumed independence of X and Y. Substituting in the corresponding densities gives

$$f_{U,V}(u,v) = |v| \cdot \frac{1}{\sqrt{2\pi}} e^{-u^2v^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-v^2/2} = \frac{|v|}{2\pi} e^{-\frac{v^2}{2}(u^2+1)}$$

provided $-\infty < u, v < \infty$. The marginal density of U is

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,\mathrm{d}v = \int_{-\infty}^{\infty} \frac{|v|}{2\pi} e^{-\frac{v^2}{2}(u^2+1)}\,\mathrm{d}v = 2\int_0^{\infty} \frac{v}{2\pi} e^{-\frac{v^2}{2}(u^2+1)}\,\mathrm{d}v$$
$$= \int_0^{\infty} \frac{v}{\pi} e^{-\frac{v^2}{2}(u^2+1)}\,\mathrm{d}v$$

since the integrand is even. Making the substitution $z = -v^2(u^2+1)/2$ so that $\mathrm{d}z = -v(u^2+1)\,\mathrm{d}v$ gives

$$f_U(u) = \frac{1}{\pi(u^2+1)} \int_0^{\infty} e^{-z}\,\mathrm{d}z = \frac{1}{\pi(u^2+1)}$$

for $-\infty < u < \infty$. We recognize that this is the density of a $C(0,1)$ random variable, and so we conclude that $U = X/Y \in C(0,1)$.

Example (Chapter 1, Problem #39)**.** Suppose that $X_1 \in \Gamma(a_1, b)$ and $X_2 \in \Gamma(a_2, b)$ are independent random variables. Show that X_1/X_2 and X_1+X_2 are independent, and determine their distributions.

Solution. Since X_1 and X_2 are independent, their joint density is

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)\cdot f_{X_2}(x_2)$$
$$= \begin{cases} \frac{1}{\Gamma(a_1)\Gamma(a_2)} x_1^{a_1-1} x_2^{a_2-1} \frac{1}{b^{a_1+a_2}} e^{-x_1/b - x_2/b}, & \text{for } x_1>0,\ x_2>0, \\ 0, & \text{otherwise.} \end{cases}$$

Let $U = X_1/X_2$ and $V = X_1 + X_2$ so that solving for X_1 and X_2 gives

$$X_1 = \frac{UV}{U+1} \quad \text{and} \quad X_2 = \frac{V}{U+1}.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial u} & \dfrac{\partial x_1}{\partial v} \\ \dfrac{\partial x_2}{\partial u} & \dfrac{\partial x_2}{\partial v} \end{vmatrix} = \begin{vmatrix} v(1+u)^{-2} & u(1+u)^{-1} \\ -v(1+u)^{-2} & (1+u)^{-1} \end{vmatrix} = \frac{v}{(1+u)^2}.$$

The density of $(U,V)'$ is therefore given by

$$\begin{aligned} f_{U,V}(u,v) &= f_{X_1,X_2}(uv(1+u)^{-1}, v(1+u)^{-1})\cdot |J| \\ &= \frac{1}{\Gamma(a_1)\Gamma(a_2)}(uv(1+u)^{-1})^{a_1-1}(v(1+u)^{-1})^{a_2-1}\frac{1}{b^{a_1+a_2}}e^{-uv(1+u)^{-1}/b - v(1+u)^{-1}/b}\cdot\frac{|v|}{(1+u)^2} \\ &= \frac{1}{\Gamma(a_1)\Gamma(a_2)}\frac{1}{b^{a_1+a_2}}\frac{u^{a_1-1}v^{a_1+a_2-1}}{(1+u)^{a_1+a_2}}e^{-v/b} \end{aligned}$$

provided that $0 < u < \infty$, $0 < v < \infty$. The marginal density of U is

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,\mathrm{d}v = \frac{1}{\Gamma(a_1)\Gamma(a_2)}\frac{1}{b^{a_1+a_2}}\frac{u^{a_1-1}}{(1+u)^{a_1+a_2}}\int_0^\infty v^{a_1+a_2-1}e^{-v/b}\,\mathrm{d}v$$

To evaluate

$$\int_0^\infty v^{a_1+a_2-1}e^{-v/b}\,\mathrm{d}v$$

we make the substitution $z = \frac{v}{b}$ so that $\mathrm{d}z = \frac{1}{b}\,\mathrm{d}v$. This implies that

$$\begin{aligned} \int_0^\infty v^{a_1+a_2-1}e^{-v/b}\,\mathrm{d}v = \int_0^\infty (bz)^{a_1+a_2-1}e^{-z}b\,\mathrm{d}z &= b^{a_1+a_2}\int_0^\infty z^{a_1+a_2-1}e^{-z}\,\mathrm{d}z \\ &= b^{a_1+a_2}\Gamma(a_1+a_2). \end{aligned}$$

This now implies that

$$f_U(u) = \frac{1}{\Gamma(a_1)\Gamma(a_2)}\frac{1}{b^{a_1+a_2}}\frac{u^{a_1-1}}{(1+u)^{a_1+a_2}}b^{a_1+a_2}\Gamma(a_1+a_2) = \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\frac{u^{a_1-1}}{(1+u)^{a_1+a_2}}, \quad u>0.$$

To find the marginal density of V we observe that since we can write the joint density as a product of a function of u only multiplied by a function of v only, we conclude that U and V are independent. That is,

$$f_{U,V}(u,v) = f_U(u)\cdot f_V(v)$$

and so using the density $f_U(u)$ that we just found gives

$$f_V(v) = \frac{f_{U,V}(u,v)}{f_U(u)} = \frac{\frac{1}{\Gamma(a_1)\Gamma(a_2)}\frac{1}{b^{a_1+a_2}}\frac{u^{a_1-1}v^{a_1+a_2-1}}{(1+u)^{a_1+a_2}}e^{-v/b}}{\frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\frac{u^{a_1-1}}{(1+u)^{a_1+a_2}}} = \frac{1}{\Gamma(a_1+a_2)}v^{a_1+a_2-1}\frac{1}{b^{a_1+a_2}}e^{-v/b}$$

for $v > 0$. Notice that $V = X_1 + X_2 \in \Gamma(a_1 + a_2, b)$.

It is also possible to find the marginal density of V by integrating the joint density. That is, we observe that

$$f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,\mathrm{d}u = \frac{1}{\Gamma(a_1)\Gamma(a_2)}\frac{1}{b^{a_1+a_2}}v^{a_1+a_2-1}e^{-v/b}\int_0^{\infty}\frac{u^{a_1-1}}{(1+u)^{a_1+a_2}}\,\mathrm{d}u.$$

Making the substitution $z = u/(1+u)$ so that $\mathrm{d}z = (1+u)^{-2}\,\mathrm{d}u$ and $u = z/(1-z)$ implies

$$\begin{aligned}\int_0^{\infty}\frac{u^{a_1-1}}{(1+u)^{a_1+a_2}}\,\mathrm{d}u = \int_0^{\infty}\left(\frac{u}{1+u}\right)^{a_1+a_2-2}\cdot u^{1-a_2}\cdot\frac{1}{(1+u)^2}\,\mathrm{d}u &= \int_0^1 z^{a_1+a_2-2}\left(\frac{z}{1-z}\right)^{1-a_2}\mathrm{d}z \\ &= \int_0^1 z^{a_1-1}(1-z)^{a_2-1}\,\mathrm{d}z.\end{aligned}$$

We recognize this as a beta integral so that

$$\int_0^1 z^{a_1-1}(1-z)^{a_2-1}\,\mathrm{d}z = \frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(a_1+a_2)}$$

from which we conclude

$$f_V(v) = \frac{1}{\Gamma(a_1+a_2)}\frac{1}{b^{a_1+a_2}}v^{a_1+a_2-1}e^{-v/b}$$

for $v > 0$ as before.