Lecture #6: The Jacobian for Polar Coordinates

Recall. Suppose that $\mathbf{X}$ is a random vector with joint density function $f_{\mathbf{X}}(\overline{x})$. If we define the random vector $\mathbf{Y} = g(\mathbf{X})$, then we proved last lecture that the density for $\mathbf{Y}$ is given by

$$f_{\mathbf{Y}}(\overline{y}) = f_{\mathbf{X}}(h(\overline{y})) \cdot |J| \qquad (\dagger)$$

where $h = g^{-1}$ so that $\mathbf{X} = g^{-1}(\mathbf{Y}) = h(\mathbf{Y})$, and J is the Jacobian.

Example. Let $X, Y \in \text{Exp}(1)$ be independent. Prove that $\frac{X}{X+Y}$ and $X+Y$ are independent random variables, and determine their distributions.

Solution. Let $U = \frac{X}{X+Y}$ and $V = X + Y$ so that

$$X = UV \quad \text{and} \quad Y = V - UV.$$

We compute the Jacobian of this transformation as

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = v.$$

By the result (†) from Lecture #4, we conclude

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}(uv, v - uv) \cdot v \\ &= f_X(uv) \cdot f_Y(v - uv) \cdot v \quad \text{by the assumed independence of } X \text{ and } Y \\ &= e^{-uv} \cdot e^{-v+uv} \cdot v \\ &= ve^{-v} \end{aligned}$$

for $0 < u < 1$, $v > 0$. In other words,

$$f_{U,V}(u,v) = \begin{cases} 1 \cdot ve^{-v}, & \text{if } 0 < u < 1 \text{ and } v > 0, \\ 0, & \text{otherwise.} \end{cases}$$

That is, since $f_{U,V}(u,v) = f_U(u) \cdot f_V(v)$, we see that $U \in U(0,1)$ and $V \in \Gamma(2,1)$ are independent random variables.

Example. Determine the Jacobian for the change-of-variables from cartesian coordinates to polar coordinates.

Solution. The traditional letters to use are

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

However, to agree with the notation from class, we let

$$x = u\cos v \quad \text{and} \quad y = u\sin v.$$

In other words, our original variables are $\overline{x} = (x, y)$, and our new variables are $\overline{y} = (u, v)$. The variables $\overline{x}$ and $\overline{y}$ are related through the function $g : \mathbb{R}^2 \to \mathbb{R}^2$ defined implicitly by

$$g(\overline{x}) = g(x, y) = (g_1(x, y), g_2(x, y)) = (u, v).$$

In other words,

$$u = \sqrt{x^2 + y^2} \quad \text{and} \quad v = \arctan(y/x).$$

We now compute the required partial derivatives:

$$\frac{\partial x}{\partial u} = \cos v\ ,\ \ \frac{\partial x}{\partial v} = -u\sin v\ ,\ \ \frac{\partial y}{\partial u} = \sin v\ ,\ \ \frac{\partial y}{\partial v} = u\cos v.$$

Therefore, the Jacobian is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix} = u\cos^2 v + u\sin^2 v = u.$$

Thus, using the traditional notation, $|J| = r$. i.e., $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$.

Exercise. Consider the three-dimensional change of variables to cylindrical coordinates given by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

Compute the Jacobian of this transformation and show that $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$.

Exercise. Consider the three-dimensional change of variables to spherical coordinates given by

$$x = \rho\cos\theta\sin\varphi, \quad y = \rho\sin\theta\sin\varphi, \quad z = \rho\cos\varphi.$$

Compute the Jacobian of this transformation and show that $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$.