Make sure that this examination has 12 numbered pages

University of Regina
Department of Mathematics & Statistics
Final Examination
200930
(December 16, 2009)

Statistics 351
Intermediate Probability

Name: ______________________ **Student Number:** ______________

Instructor: Michael Kozdron **Time:** 3 hours

Read all of the following information before starting the exam.

*You have **3** hours to complete this exam. Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit. **Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.***

You may use standard notation; however, any new notations or abbreviations that you introduce must be clearly defined.

*Calculators are permitted; however, you must still show all your work. You are also permitted to have **TWO** 8.5×11 pages of handwritten notes (double-sided) for your personal use. Other than these exceptions, no other aids are allowed.*

Note that blank space is not an indication of a question's difficulty. The order of the test questions is essentially random; they are not intentionally written easiest-to-hardest.

*This test has **12** numbered pages with **11** questions totalling **150** points. The number of points per question is indicated. For questions with multiple parts, all parts are equally weighted.*

Fact: For $\lambda > 0$, the density function of a random variable $X \in \text{Exp}(\lambda)$ is

$$f_X(x) = \frac{1}{\lambda}\, e^{-x/\lambda}, \quad 0 < x < \infty.$$

Fact: For $p > 0$, the Gamma function is given by

$$\Gamma(p) = \int_0^\infty x^{p-1}\, e^{-x}\, \mathrm{d}x.$$

Fact: You may find it useful to know that $\cos(2x) = 1 - 2\sin^2(x)$.

DO NOT WRITE BELOW THIS LINE

Problem 1	______	Problem 2	______	Problem 3	______
Problem 4	______	Problem 5	______	Problem 6	______
Problem 7	______	Problem 8	______	Problem 9	______
Problem 10	______	Problem 11	______		
				TOTAL	______

1. (*28 points*) Suppose that a random vector $(X, Y)'$ has joint density function

$$f_{X,Y}(x,y) = \begin{cases} 15x^2y, & \text{if } 0 < x < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Verify that $f_{X,Y}$ is, in fact, a legitimate density function.

(b) Find $f_X(x)$, the marginal density function of X.

(c) Use your result of **(b)** to compute $E(X)$.

(d) Find $f_{Y|X=x}(y)$, the conditional density function of $Y|X = x$.

(e) Compute $E(Y|X = x)$.

(continued)

(f) Use your results of **(b)** and **(e)** to compute $E(Y)$.

(g) Compute $\text{Cov}(X, Y)$.

2. (*12 points*) Suppose that a random vector $(X, Y)'$ has joint density function

$$f_{X,Y}(x,y) = \begin{cases} 2(1-x), & \text{if } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Define the random vector $(U, V)'$ by setting $U = XY$ and $V = X$.

(a) Determine $f_{U,V}(u,v)$, the joint density function of $(U, V)'$.

(b) Determine $f_U(u)$, the marginal density function of U.

3. (*12 points*)

(a) Suppose that the random variable $Z \in \Gamma(a, 1/b)$ with $a > 0$ and $b > 0$ so that the density function of Z is

$$f_Z(z) = \frac{b^a}{\Gamma(a)} z^{a-1} e^{-bz}, \quad z > 0.$$

Determine the density function of the random variable $X = \dfrac{1}{Z}$. We say that X has an *inverse Gamma distribution with parameters a and b.*

(b) Let $(X, Y)'$ be jointly distributed, and suppose that the conditional distribution of Y given $X = x$ is normal with mean 0 and variance $\frac{x}{2}$; that is, $Y|X = x \in N(0, \frac{x}{2})$. Thus, the conditional density function of $Y|X = x$ is

$$f_{Y|X=x}(y) = \frac{1}{\sqrt{x\pi}} e^{-y^2/x}, \quad -\infty < y < \infty.$$

If the random variable X has an inverse Gamma distribution with parameters a and b so that its density function is what you found in **(a)**, determine an exact, closed-form expression for $f_Y(y)$, the marginal density function of Y.

4. (*8 points*) Let α, $\beta \in \mathbb{R}$ be unknown real numbers, and suppose that $(X, Y)'$ is a two-dimensional random vector. It is known that the marginal distribution of X has mean α and variance 2, and that the marginal distribution of Y has mean 4 and variance 23. It is also known that $E(Y|X) = 2X$ and $\text{Var}(Y|X) = X^2 + \beta$.

(a) Determine the value of α.

(b) Using the value of α that you found in **(a)**, determine the value of β.

5. (*8 points*) The purpose of this problem is for you to derive a result for order statistics of random variables which are independent, but are not identically distributed. Let $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$ be three distinct positive numbers, and suppose that $X_1 \in \text{Exp}(\lambda_1)$, $X_2 \in \text{Exp}(\lambda_2)$, and $X_3 \in \text{Exp}(\lambda_3)$ are independent. Determine the density function of $Y = \min\{X_1, X_2, X_3\}$.

6. (*12 points*) Suppose that $a > 1$, and let X_1, X_2, X_3, X_4 be independent and identically distributed $U(0, a)$ random variables so that their common density function is

$$f(x) = \begin{cases} \dfrac{1}{a}, & \text{if } 0 < x < a, \\ 0, & \text{otherwise.} \end{cases}$$

As always, let $X_{(j)}$, $j = 1, 2, 3, 4$, denote the jth order variable.

(a) Determine the joint density function of $(X_{(2)}, X_{(3)})'$.

(b) Compute $P(X_{(3)} < aX_{(2)})$.

7. (*18 points*) Suppose that $\mathbf{X} = (X_1, X_2, X_3)' \in N(\boldsymbol{\mu}, \mathbf{\Lambda})$ where

$$\boldsymbol{\mu} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad \text{and} \quad \mathbf{\Lambda} = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Define the random vector $\mathbf{Y} = (Y_1, Y_2)'$ by settting

$$\begin{aligned} Y_1 &= X_1 + X_2 + X_3, \quad \text{and} \\ Y_2 &= X_1 - X_2. \end{aligned}$$

(a) Determine the distribution of $\mathbf{Y}$.

(b) Determine $f_{\mathbf{Y}}(y_1, y_2)$, the density function of $\mathbf{Y}$.

(c) Determine the conditional distribution of Y_2 given $Y_1 = 0$.

8. (*16 points*) Suppose that the random vector $\mathbf{X} = (X_1, X_2)'$ has the multivariate normal distribution $\mathbf{X} \in N(\boldsymbol{\mu}, \boldsymbol{\Lambda})$ where

$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Lambda} = \begin{bmatrix} \frac{7}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{5}{4} \end{bmatrix}.$$

(a) Determine the eigenvalues of $\boldsymbol{\Lambda}$.

(b) Find an orthogonal matrix C and a diagonal matrix D such that $CDC' = \boldsymbol{\Lambda}$.

(continued)

(c) Define the random vector $\mathbf{Y} = (Y_1, Y_2)'$ by setting $\mathbf{Y} = C'\mathbf{X}$ where C is the orthogonal matrix you found in **(b)**. Determine the distribution of $\mathbf{Y}$.

(d) Explain why the random variables Y_1 and Y_2 you found in **(c)** are independent.

9. (*12 points*) Let $\mathbf{X} = (X_1, X_2)' \in N(\boldsymbol{\mu}, \boldsymbol{\Lambda})$ where

$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Lambda} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.$$

(a) Show that

$$Y_1 = \frac{X_1 - \rho X_2}{\sqrt{1-\rho^2}} \quad \text{and} \quad Y_2 = X_2$$

are independent $N(0,1)$ random variables.

(b) Determine the distribution of $\dfrac{X_1^2 - 2\rho X_1 X_2 + X_2^2}{1-\rho^2}$. *Hint:* Consider $Y_1^2 + Y_2^2$.

10. (*12 points*) Suppose that the random vector $\mathbf{X} = (X_1, X_2)'$ has the multivariate normal distribution

$$\mathbf{X} \in N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \right),$$

and that the random vector $\mathbf{Y} = (Y_1, Y_2)'$ has the multivariate normal distribution

$$\mathbf{Y} \in N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \right),$$

where $0 < \rho < 1$. Suppose further that $\mathbf{X}$ and $\mathbf{Y}$ are independent, and let the random vector $\mathbf{Z} = (Z_1, Z_2)'$ be defined as $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$ so that $Z_1 = X_1 + Y_1$ and $Z_2 = X_2 + Y_2$. Carefully verify that

$$\mathbf{Z} \in N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right).$$

Hint: Solve this problem in three steps. Begin by first computing $E(\mathbf{Z})$. Then compute $\mathrm{Cov}(\mathbf{Z})$. *Finally, use Definition I and other facts about normal random variables to conclude that* $\mathbf{Z}$ *has a MVN distribution.*

11. (*12 points*) Suppose that X_1 and X_2 are independent and identically distributed random variables with common density function

$$f(x) = \frac{\sqrt{2}}{\sqrt{\pi}}\, e^{-x^2/2}, \quad x > 0.$$

Let $Y = \min\{X_1, X_2\}$.

(a) Determine an expression for the density function of Y. *Note that your answer will need to be written in terms of an integral that you cannot evaluate explicitly.*

(b) Use your answer from **(a)** to write down an expression that respresents $E(Y^2)$. *Note that your expression will be written in terms of a double integral.*

(c) Compute $E(Y^2)$ by evaluating the double integral you wrote down in **(b)**. *You may need to switch to polar coordinates in order to perform the computation.*

The End. Happy Holidays!