Make sure that this examination has 11 numbered pages

University of Regina
Department of Mathematics & Statistics

Final Examination
200730
(December 14, 2007)

Statistics 351
Probability I

Name: ______________________________ **Student Number:** ______________

Instructor: Michael Kozdron **Time:** 3 hours

Read all of the following information before starting the exam.

*You have **3** hours to complete this exam. Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit.* ***Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.***

You may use standard notation; however, any new notations or abbreviations that you introduce must be clearly defined.

*Calculators are permitted; however, you must still show all your work. You are also permitted to have **TWO** 8.5×11 pages of handwritten notes (double-sided) for your personal use. Other than these exceptions, no other aids are allowed.*

Note that blank space is not an indication of a question's difficulty. The order of the test questions is essentially random; they are not intentionally written easiest-to-hardest.

*This test has **11** numbered pages with **11** questions totalling **150** points. The number of points per question is indicated.*

Fact: For $\lambda > 0$, the density of a random variable $X \in \text{Exp}(\lambda)$ is

$$f_X(x) = \frac{1}{\lambda} \exp\left\{-\frac{x}{\lambda}\right\}, \quad 0 < x < \infty.$$

Fact: For $p > 0$, the Gamma function is given by

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx.$$

DO NOT WRITE BELOW THIS LINE

Problem 1	________	Problem 2	________	Problem 3	________
Problem 4	________	Problem 5	________	Problem 6	________
Problem 7	________	Problem 8	________	Problem 9	________
Problem 10	________	Problem 11	________		
				TOTAL	________

1. (*24 points*) Suppose that a random vector $(X, Y)'$ has joint density function

$$f_{X,Y}(x,y) = \begin{cases} 8xy, & \text{if } 0 < x < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Verify that $f_{X,Y}$ is, in fact, a density.

(b) Find $f_X(x)$, the marginal density function of X.

(c) Use your result of **(b)** to compute $E(X)$.

(d) Find $f_{Y|X=x}(y)$, the conditional density function of $Y|X = x$.

(e) Compute $E(Y|X = x)$.

(f) Use your results of **(b)** and **(e)** to compute $E(Y)$.

2. (*12 points*) Let $\mathbf{X} = (X_1, X_2, X_3)'$ have the multivariate normal distribution

$$\mathbf{X} \in N\left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}2 & 0 & -1\\0 & 1 & -1\\-1 & -1 & 2\end{pmatrix}\right).$$

Let the random vector $\mathbf{Y} = (Y_1, Y_2)'$ be defined by $Y_1 = X_1 + 2X_2 - X_3$ and $Y_2 = X_2 - X_3$.

(a) Determine the distribution of $\mathbf{Y}$.

(b) Determine $f_{Y_1,Y_2}(y_1, y_2)$, the density function of $\mathbf{Y}$.

(c) Determine $\varphi(t_1, t_2)$, the characteristic function of $\mathbf{Y}$.

3. (*16 points*) Let $\mathbf{X} = (X_1, X_2)'$ be multivariate normal with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Lambda}$ where

$$\boldsymbol{\mu} = \begin{pmatrix} 5 \\ \beta \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Lambda} = \begin{pmatrix} 1 & \alpha \\ \alpha & 25 \end{pmatrix}$$

(a) If $\mathrm{Var}(X_2|X_1) = 16$, determine all possible values of α.

(b) Suppose that $E(X_2|X_1 = 6) = 1$. Based on your answer to **(a)**, determine all possible values of β.

4. (*24 points*) Suppose that the random vector $\mathbf{X} = (X_1, X_2)'$ has the multivariate normal distribution $\mathbf{X} \in N(\boldsymbol{\mu}, \boldsymbol{\Lambda})$ where

$$\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Lambda} = \begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix}.$$

(a) Determine the eigenvalues of $\boldsymbol{\Lambda}$.

(b) Find an orthogonal matrix C and a diagonal matrix D such that $C'\boldsymbol{\Lambda}C = D$.

(c) Determine the distribution of $\mathbf{Y} = (Y_1, Y_2)'$ where $\mathbf{Y} = C'\mathbf{X}$. *Hint*: Use **(b)**.

(d) Explain why the random variables Y_1 and Y_2 are independent.

5. (*10 points*) Suppose that Y is a random variable with density function $f_Y(y) = 20y^3(1-y)$, $0 < y < 1$. Suppose further that X is a continuous random variable such that the conditional distribution of X given $Y = y$ is $U(0, y)$. That is, $X|Y = y \in U(0, y)$. Determine the marginal density function of X.

6. (*10 points*) Suppose that X and Y are continuous random variables. Suppose further that $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are strictly increasing, differentiable functions. Use the multidimensional change-of-variables theorem to show that if X and Y are independent, then $g(X)$ and $h(Y)$ are independent.

7. (*12 points*) Suppose that $\mathbf{X} = (X_1, X_2)' \in \mathcal{N}(\bar{0}, \mathbf{\Lambda})$ where $\det[\mathbf{\Lambda}] > 0$. If $f_{\mathbf{X}}(x_1, x_2)$ denotes the density function of $\mathbf{X}$, verify that

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{\mathbf{X}}(x_1, x_2) \log\left(f_{\mathbf{X}}(x_1, x_2)\right) \, \mathrm{d}x_1 \, \mathrm{d}x_2 = 1 + \log(2\pi) + \frac{1}{2}\log(\det[\mathbf{\Lambda}]).$$

This quantity is called the differential entropy of $\mathbf{X}$.

8. (*8 points*) Let $Y_1, Y_2, \ldots$ be independent and identically distributed with $P(Y_1 = 1) = P(Y_1 = -1) = 1/2$. Set $S_0 = 0$ and for $n = 1, 2, 3, \ldots$ define S_n to be

$$S_n = \sum_{i=1}^{n} Y_i.$$

Recall that $\{S_n, n = 0, 1, 2, \ldots\}$ is called a simple random walk. We showed in class that $\{S_n, n = 0, 1, 2, \ldots\}$ is a martingale and that $\mathbb{E}(S_n^2) = n$ for all $n = 0, 1, 2, \ldots$. Now suppose that $X_0 = 0$ and for $j = 1, 2, \ldots$ define X_j to be

$$X_j = \sum_{n=1}^{j} S_{n-1}(S_n - S_{n-1}).$$

Prove that $\{X_j, j = 0, 1, 2, \ldots\}$ is a martingale.

Note: It is equivalent to show $E(X_{j+1}|X_j) = X_j$ *or* $E(X_{j+1}|S_j) = X_j$ *or* $E(X_{j+1}|Y_j) = X_j$.

The process $\{X_j, j = 0, 1, 2, \ldots\}$ *is an example of a discrete stochastic integral. Stochastic integration is one of the greatest achievements of 20th century probability and is fundamental to the mathematical theory of finance and option pricing.*

9. (*12 points*) Suppose that X_1, X_2, X_3, X_4 are independent and identically distributed $U(0,1)$ random variables. Let $X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)}$ denote the order variables. Show that

$$\left(\frac{X_{(2)}}{X_{(3)}}\right)^2,$$

has a $U(0,1)$ distribution.

Hint: Let $U = \frac{X_{(2)}}{X_{(3)}}$ *and determine the density of* U*. Then determine the density of* $W = U^2$*.*

10. (*12 points*) Suppose that $\{X_t,\ t \geq 0\}$ is a Poisson process with intensity $\lambda = 1$. Evaluate the following:

(a) $P(X_5 = j)$ for $j = 1, 2$;

(b) $\text{Var}(X_5 | X_2 = 1)$;

(c) $\text{Cov}(X_2, X_4)$;

(d) $E(X_4 | X_2 = j)$ for $j = 0, 1, 2, \ldots$.

11. (*10 points*) Jessica is a compulsive shoe shopper, and she buys pairs of shoes according to a Poisson process with a rate (or intensity) of 1 pair per week.

(a) How many pairs of shoes is Jessica expected to buy in one year? (Recall that there are 52 weeks in a year.)

(b) Suppose that Jessica bought 8 pairs of shoes during the four weeks of February 2007. What is the probability that she bought 3 of them during the first week of February 2007?

The End. Happy Holidays!