Stat 351 Fall 2015
Chapter 1 Solutions

Problem #2. Suppose that $X \in C(m, a)$ so that the density of X is given by

$$f_X(x) = \frac{1}{\pi} \cdot \frac{a}{a^2 + (x-m)^2}, \quad -\infty < x < \infty.$$

Let $Y = 1/X$. If $X \neq 0$, then for $-\infty < y < \infty$ the distribution function of Y is

$$F_Y(y) = P(Y \le y) = P(1/X \le y) = P(X \ge 1/y) = 1 - P(X \le 1/y) = 1 - \int_{-\infty}^{1/y} \frac{1}{\pi} \cdot \frac{a}{a^2 + (x-m)^2}\, dx$$

so that the density function of Y is

$$f_Y(y) = \frac{1}{\pi} \cdot \frac{a}{a^2 + (1/y - m)^2} \cdot \frac{1}{y^2} = \frac{1}{\pi} \cdot \frac{a}{y^2 a^2 + (1 - ym)^2}$$

for $-\infty < y < \infty$. The only potential trouble is if $X = 0$ since $Y = 1/X$ is not defined there. However, since $P(X = 0) = 0$, this is not a problem. Thus,

$$f_Y(y) = \frac{1}{\pi} \cdot \frac{a}{y^2 a^2 + (1 - ym)^2}, \quad -\infty < y < \infty.$$

Problem #1. This is a special case of Problem #2. If $m = 0$ and $a = 1$, then

$$f_Y(y) = \frac{1}{\pi} \cdot \frac{1}{y^2 + 1}, \quad -\infty < y < \infty,$$

so that $Y = 1/X \in C(0, 1)$.

Problem #3. Suppose that $T \in t(n)$ so that the density of T is given by

$$f_T(x) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\,\Gamma(\frac{n}{2})} \cdot \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2}, \quad -\infty < x < \infty.$$

Let $Y = T^2$. If $y \ge 0$, then the distribution function of Y is given by

$$\begin{aligned} F_Y(y) = P(Y \le y) = P(T^2 \le y) = P(-\sqrt{y} \le T \le \sqrt{y}) &= \int_{-\sqrt{y}}^{\sqrt{y}} f_T(x)\, dx \\ &= \int_0^{\sqrt{y}} f_T(x)\, dx - \int_0^{-\sqrt{y}} f_T(x)\, dx. \end{aligned}$$

Taking derivatives with respect to y gives

$$\begin{aligned} f_Y(y) = f_T(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f_T(-\sqrt{y}) \cdot \frac{-1}{2\sqrt{y}} &= \frac{1}{2\sqrt{y}} \left(f_T(\sqrt{y}) + f_T(-\sqrt{y}) \right) \\ &= \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n y}\,\Gamma(\frac{n}{2})} \cdot \left(1 + \frac{y}{n}\right)^{-(n+1)/2} \\ &= \frac{\Gamma(\frac{1+n}{2}) \left(\frac{1}{n}\right)^{1/2}}{\Gamma(\frac{1}{2})\,\Gamma(\frac{n}{2})} \cdot \frac{y^{1/2-1}}{\left(1 + \frac{y}{n}\right)^{(1+n)/2}}, \quad y \ge 0. \end{aligned}$$

In order to write this last line, we have used the fact that $\Gamma(1/2) = \sqrt{\pi}$. Notice that this is the density of an $F(1, n)$ random variable.

Problem #4. Suppose that $X \in F(m,n)$ so that the density of X is given by

$$f_X(x) = \frac{\Gamma(\frac{m+n}{2})\left(\frac{m}{n}\right)^{n/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \frac{x^{m/2-1}}{(1+\frac{mx}{n})^{(m+n)/2}}, \quad 0 < x < \infty.$$

Let $Y = 1/X$ so that for $0 < y < \infty$ the distribution function of Y is

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(1/X \le y) = P(X \ge 1/y) = 1 - P(X \le 1/y) \\ &= 1 - \int_{-\infty}^{1/y} \frac{\Gamma(\frac{m+n}{2})\left(\frac{m}{n}\right)^{n/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \frac{x^{m/2-1}}{(1+\frac{mx}{n})^{(m+n)/2}}\,dx. \end{aligned}$$

Taking derivatives with respect to y gives

$$\begin{aligned} f_Y(y) &= \frac{\Gamma(\frac{m+n}{2})\left(\frac{m}{n}\right)^{n/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \frac{y^{1-m/2}}{(1+\frac{m}{ny})^{(m+n)/2}} \cdot \frac{1}{y^2} \\ &= \frac{\Gamma(\frac{m+n}{2})\left(\frac{m}{n}\right)^{n/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \frac{y^{1-m/2}}{(\frac{m}{ny})^{(m+n)/2}(1+\frac{ny}{m})^{(m+n)/2}} \cdot \frac{1}{y^2} \\ &= \frac{\Gamma(\frac{m+n}{2})\left(\frac{n}{m}\right)^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \frac{y^{n/2-1}}{(1+\frac{ny}{m})^{(m+n)/2}} \end{aligned}$$

for $y > 0$. We recognize that this is the density of a $F(n,m)$ random variable, and so we conclude that $Y = 1/X \in F(n,m)$.

Problem #5. Suppose that $X \in C(0,1)$ so that the density of X is given by

$$f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}, \quad -\infty < x < \infty.$$

Let $Y = X^2$. If $y \ge 0$, then the distribution function of Y is given by

$$\begin{aligned} F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) &= \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx \\ &= \int_0^{\sqrt{y}} f_X(x)\,dx - \int_0^{-\sqrt{y}} f_X(x)\,dx. \end{aligned}$$

Taking derivatives with respect to y gives

$$f_Y(y) = f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f_X(-\sqrt{y}) \cdot \frac{-1}{2\sqrt{y}} = \frac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right) = \frac{1}{\pi\sqrt{y}} \cdot \frac{1}{1+y}, \quad y \ge 0.$$

Notice that this is the density of an $F(1,1)$ random variable. (Recall that $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$.)

Problem #6. If $X \in \beta(1,1)$, then the density function of X is

$$f_X(x) = \frac{\Gamma(1+1)}{\Gamma(1)\Gamma(1)} x^{1-1}(1-x)^{1-1} = 1, \quad 0 < x < 1.$$

(We have used the fact that $\Gamma(2) = \Gamma(1) = 1$.) Since the density of X is also that of a uniform random variable, we conclude $X \in U(0,1)$. Therefore, $\beta(1,1) = U(0,1)$.

Problem #7. Suppose that $X \in F(m,n)$ so that the density of X is given by

$$f_X(x) = \frac{\Gamma(\frac{m+n}{2})\left(\frac{m}{n}\right)^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \frac{x^{m/2-1}}{(1+\frac{mx}{n})^{(m+n)/2}}, \quad 0 < x < \infty.$$

Let $Y = 1/(1+\frac{m}{n}X)$ so that if $0 \le y \le 1$, then the distribution function of Y is

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(1/(1+\tfrac{m}{n}X) \le y) = P(1+\tfrac{m}{n}X \ge 1/y) = P(X \ge \tfrac{n}{m}(1/y-1)) \\ &= 1 - P(X \le \tfrac{n}{m}(1/y-1)) \\ &= 1 - \int_0^{\frac{n}{m}(1/y-1)} \frac{\Gamma(\frac{m+n}{2})\left(\frac{m}{n}\right)^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \frac{x^{m/2-1}}{(1+\frac{mx}{n})^{(m+n)/2}}\, dx. \end{aligned}$$

Taking derivatives with respect to y gives

$$\begin{aligned} f_Y(y) &= \frac{\Gamma(\frac{m+n}{2})\left(\frac{m}{n}\right)^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \frac{(\frac{n}{m}(1/y-1))^{m/2-1}}{(1+\frac{m\frac{n}{m}(1/y-1)}{n})^{(m+n)/2}} \cdot \frac{n}{my^2} \\ &= \frac{\Gamma(\frac{m+n}{2})\left(\frac{m}{n}\right)^{m/2}\left(\frac{n}{m}\right)^{m/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \frac{(1/y-1)^{m/2-1}}{(1/y)^{(m+n)/2}} \cdot \frac{1}{y^2} \\ &= \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} y^{m/2+n/2} y^{-2} y^{1-m/2}(1-y)^{m/2-1} \\ &= \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} y^{n/2-1}(1-y)^{m/2-1} \end{aligned}$$

for $0 \le y \le 1$. We recognize that this is the density of a $\beta(n/2, m/2)$ random variable, and so we conclude that $Y = 1/(1+\frac{m}{n}X) \in \beta(n/2, m/2)$.

Problem #8. Suppose that $X \in N(0,1)$ and $Y \in N(0,1)$ are independent random variables. Let $U = \frac{X}{Y}$ and $V = Y$ so that solving for X and Y gives

$$X = UV \quad \text{and} \quad Y = V.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v.$$

The density of $(U,V)'$ is therefore given by

$$f_{U,V}(u,v) = f_{X,Y}(uv, v) \cdot |J| = |v| f_X(uv) f_Y(v)$$

using the assumed independence of X and Y. Substituting in the corresponding densities gives

$$f_{U,V}(u,v) = |v| \cdot \frac{1}{\sqrt{2\pi}} e^{-u^2v^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-v^2/2} = \frac{|v|}{2\pi} e^{-\frac{v^2}{2}(u^2+1)}$$

provided $-\infty < u, v < \infty$. The marginal density of U is

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\, dv = \int_{-\infty}^{\infty} \frac{|v|}{2\pi} e^{-\frac{v^2}{2}(u^2+1)}\, dv = 2\int_0^{\infty} \frac{v}{2\pi} e^{-\frac{v^2}{2}(u^2+1)}\, dv = \int_0^{\infty} \frac{v}{\pi} e^{-\frac{v^2}{2}(u^2+1)}\, dv$$

since the integrand is even.

(continued)

Making the substitution $z = -v^2(u^2+1)/2$ so that $dz = -v(u^2+1)$ gives

$$f_U(u) = \frac{1}{\pi(u^2+1)} \int_0^\infty e^{-z}\, dz = \frac{1}{\pi(u^2+1)}$$

for $-\infty < u < \infty$. We recognize that this is the density of a $C(0,1)$ random variable, and so we conclude that $U = X/Y \in C(0,1)$.

Problem #9. Suppose that $X \in N(0,1)$ and $Y \in \chi^2(n)$ are independent random variables. Let $U = \frac{X}{\sqrt{Y/n}}$ and $V = \sqrt{Y/n}$ so that solving for X and Y gives

$$X = UV \quad \text{and} \quad Y = nV^2.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 2nv \end{vmatrix} = 2nv^2.$$

The density of $(U,V)'$ is therefore given by

$$f_{U,V}(u,v) = f_{X,Y}(uv, nv^2) \cdot |J| = 2nv^2 f_X(uv) f_Y(nv^2)$$

using the assumed independence of X and Y. Substituting in the corresponding densities gives

$$f_{U,V}(u,v) = 2nv^2 \frac{1}{\sqrt{2\pi}} e^{-u^2v^2/2} \frac{1}{\Gamma(n/2)} (nv^2)^{n/2-1} 2^{-n/2} e^{-nv^2/2} = \frac{n^{n/2}}{2^{n/2-1/2}\sqrt{\pi}\,\Gamma(n/2)} v^n e^{-v^2(u^2+n)/2}$$

provided that $-\infty < u < \infty$, $0 < v < \infty$. The marginal density of U is

$$f_U(u) = \int_{-\infty}^\infty f_{U,V}(u,v)\, dv = \frac{n^{n/2}}{2^{n/2-1/2}\sqrt{\pi}\,\Gamma(n/2)} \int_0^\infty v^n e^{-v^2(u^2+n)/2}\, dv.$$

Making the substitution $z = v^2(u^2+n)/2$ so that $dz = v(u^2+n)dv$ gives

$$\begin{aligned}
&\frac{n^{n/2}}{2^{n/2-1/2}\sqrt{\pi}\,\Gamma(n/2)} \int_0^\infty v^n e^{-v^2(u^2+n)/2}\, dv \\
&\qquad = \frac{n^{n/2}}{2^{n/2-1/2}\sqrt{\pi}\,\Gamma(n/2)} (u^2+n)^{-1/2-n/2} 2^{n/2-1/2} \int_0^\infty z^{n/2-1/2} e^{-z}\, dz \\
&\qquad = \frac{n^{n/2}}{\sqrt{\pi}\,\Gamma(n/2)} (u^2+n)^{-1/2-n/2} \Gamma(n/2+1/2) \\
&\qquad = \frac{n^{n/2}\Gamma(n/2+1/2)}{\sqrt{\pi}\,\Gamma(n/2)} (u^2+n)^{-1/2-n/2} \\
&\qquad = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\,\Gamma(\frac{n}{2})} \frac{1}{(1+\frac{u^2}{n})^{(n+1)/2}}
\end{aligned}$$

provided $-\infty < u < \infty$. We recognize that this is the density of a $t(n)$ random variable, and so we conclude that $U = \frac{X}{\sqrt{Y/n}} \in t(n)$.

Problem #10. Suppose that $X \in \chi^2(m)$ and $Y \in \chi^2(n)$ are independent random variables. Let $U = \frac{X/m}{Y/n}$ and $V = Y/n$ so that solving for X and Y gives

$$X = mUV \quad \text{and} \quad Y = nV.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} mv & mu \\ 0 & n \end{vmatrix} = mnv.$$

The density of $(U, V)'$ is therefore given by

$$f_{U,V}(u, v) = f_{X,Y}(muv, nv) \cdot |J| = mn|v| f_X(muv) f_Y(nv)$$

using the assumed independence of X and Y. For $0 < u < \infty$, $0 < v < \infty$, we can substitute in to the corresponding densities (noting that $|v| = v$) to conclude

$$\begin{aligned} f_{U,V}(u, v) &= mnv \frac{1}{\Gamma(m/2)} (muv)^{m/2-1} 2^{-m/2} e^{-muv/2} \frac{1}{\Gamma(n/2)} (nv)^{n/2-1} 2^{-n/2} e^{-nv/2} \\ &= \frac{m^{m/2} n^{n/2} 2^{-m/2-n/2}}{\Gamma(m/2)\Gamma(n/2)} v^{m/2+n/2-1} u^{m/2-1} e^{-v(mu+n)/2}. \end{aligned}$$

The marginal density of U is

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u, v)\, dv = \frac{m^{m/2} n^{n/2} 2^{-m/2-n/2}}{\Gamma(m/2)\Gamma(n/2)} u^{m/2-1} \int_0^{\infty} v^{m/2+n/2-1} e^{-v(mu+n)/2}\, dv.$$

Making the substitution $z = v(mu + n)/2$ so that $dz = (mu + n)/2dv$ gives

$$\begin{aligned} &\frac{m^{m/2} n^{n/2} 2^{-m/2-n/2}}{\Gamma(m/2)\Gamma(n/2)} u^{m/2-1} \int_0^{\infty} v^{m/2+n/2-1} e^{-v(mu+n)/2}\, dv \\ &\qquad = \frac{m^{m/2} n^{n/2} 2^{-m/2-n/2}}{\Gamma(m/2)\Gamma(n/2)} u^{m/2-1} 2^{m/2+n/2} (mu+n)^{-m/2-n/2} \int_0^{\infty} z^{m/2+n/2+1} e^{-z}\, dz \\ &\qquad = \frac{m^{m/2} n^{n/2}}{\Gamma(m/2)\Gamma(n/2)} u^{m/2-1} (mu+n)^{-m/2-n/2} \Gamma(m/2 + n/2) \\ &\qquad = \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} m^{m/2} n^{n/2} \frac{u^{m/2-1}}{(mu+n)^{(m+n)/2}} \\ &\qquad = \frac{\Gamma(\frac{m+n}{2}) \left(\frac{m}{n}\right)^{n/2}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \frac{u^{m/2-1}}{(1 + \frac{mu}{n})^{(m+n)/2}} \end{aligned}$$

provided $0 < u < \infty$. We recognize that this is the density of a $F(m, n)$ random variable, and so we conclude that $U = \frac{X/m}{Y/n} \in F(m, n)$.

Problem #11. If $X \in \text{Exp}(a)$, then a quick calculation shows that $\frac{2X}{a} \in \text{Exp}(2)$. However, comparing the exponential and chi-square densities, we see that $\text{Exp}(2) = \chi^2(2)$. Similarly, $2Y/a \in \text{Exp}(2) = \chi^2(2)$. Thus, using the result of Problem #10, we conclude that

$$\frac{X}{Y} = \frac{2X/a}{2Y/a} = \frac{(2X/a)/2}{(2Y/a)/2} \in F(2, 2).$$

Problem #29. Suppose that X and Y have joint density

$$f_{X,Y}(x,y) = \begin{cases} \frac{2}{(1+x+y)^3} & \text{for } x, y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Let $U = X + Y$ and $V = \frac{X}{X+Y}$ so that solving for X and Y gives

$$X = UV \quad \text{and} \quad Y = U - UV.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -u.$$

The density of $(U, V)'$ is therefore given by

$$f_{U,V}(u,v) = f_{X,Y}(uv, u - uv) \cdot |J| = \frac{2}{(1 + uv + u - uv)^3} \cdot |-u| = \frac{2u}{(1+u)^3},$$

provided that $0 < u < \infty$, $0 < v < 1$. Since we can write the joint density as a product of a function of u only multiplied by a function of v only, we conclude that

$$f_{U,V}(u,v) = f_U(u) \cdot f_V(v)$$

where

$$f_U(u) = \frac{2u}{(1+u)^3} \text{ for } u > 0, \quad \text{and} \quad f_V(v) = 1 \text{ for } 0 < v < 1.$$

Therefore, the density of $X + Y$ is

$$f_{X+Y}(u) = \frac{2u}{(1+u)^3} \text{ for } u > 0.$$

(b) Let $U = X - Y$ and $V = X$, so that solving for X and Y gives

$$X = V \quad \text{and} \quad Y = V - U.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} = 1.$$

The density of $(U, V)'$ is therefore given by

$$f_{U,V}(u,v) = f_{X,Y}(v, v - u) \cdot |J| = \frac{2}{(1 + v + v - u)^3} \cdot 1 = \frac{2}{(1 + 2v - u)^3},$$

provided that $v > u$ and $v > 0$ (i.e., $v > \max\{0, u\}$), and $-\infty < u < \infty$. If $u > 0$, then $\max\{u, 0\} = u$, and so we calculate

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\, dv = \int_u^{\infty} \frac{2}{(1+2v-u)^3}\, dv = \frac{1}{2(1+2v-u)^2}\Bigg|_u^{\infty} = \frac{1}{2(1+u)^2}.$$

If $u \le 0$, then $\max\{u, 0\} = 0$, and so we calculate

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\, dv = \int_0^{\infty} \frac{2}{(1+2v-u)^3}\, dv = \frac{1}{2(1+2v-u)^2}\Bigg|_0^{\infty} = \frac{1}{2(1-u)^2}.$$

Therefore, the density of $X - Y$ is

$$f_{X-Y}(u) = \frac{1}{2(1+|u|)^2} \text{ for } -\infty < u < \infty.$$

Problem #21. Suppose that X and Y have joint density

$$f_{X,Y}(x,y) = \begin{cases} \frac{x}{(1+x)^2(1+xy)^2}, & \text{for } x, y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let $U = X$ and $V = XY$ so that solving for X and Y gives

$$X = U \quad \text{and} \quad Y = V/U.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -v/u^2 & 1/u \end{vmatrix} = \frac{1}{u}.$$

The density of $(U,V)'$ is therefore given by

$$f_{U,V}(u,v) = f_{X,Y}(u, v/u) \cdot |J| = \frac{u}{(1+u)^2(1+u \cdot v/u)^2} \cdot \frac{1}{u} = \frac{1}{(1+u)^2} \cdot \frac{1}{(1+v)^2},$$

provided that $0 < u < \infty$, $0 < v < \infty$. Since we can write the joint density as a product of a function of u only multiplied by a function of v only, we conclude that U and V are independent. That is,

$$f_{U,V}(u,v) = f_U(u) \cdot f_V(v)$$

where

$$f_U(u) = \frac{1}{(1+u)^2} \text{ for } u > 0, \quad \text{and} \quad f_V(v) = \frac{1}{(1+v)^2} \text{ for } v > 0.$$

Notice that both U and V have the same distribution, namely $F(2,2)$.

Problem #23. Suppose that $U = X^2Y$ and let $V = X$. Solving for X and Y gives

$$X = V \quad \text{and} \quad Y = \frac{U}{V^2}.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ v^{-2} & -2uv^{-3} \end{vmatrix} = -v^{-2}.$$

If the density of $(X,Y)'$ is

$$f_{X,Y}(x,y) = \begin{cases} e^{-x^2y}, & \text{for } x \geq 1,\ y > 0, \\ 0, & \text{otherwise,} \end{cases}$$

then the density of $(U,V)'$ is therefore given by

$$f_{U,V}(u,v) = f_{X,Y}(v, uv^{-2}) \cdot |J| = \frac{1}{v^2}\, e^{-u}$$

provided that $v \geq 1$ and $u > 0$. We can now determine the density of U as follows.

Routine Way: The marginal density of U is

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,dv = \int_1^{\infty} \frac{1}{v^2}\,e^{-u}\,dv = e^{-u}\left[-v^{-1}\right]_1^{\infty} = e^{-u}$$

for $u > 0$. We recognize that this is the density of an exponential random variable with parameter 1; that is, $U = X^2Y \in \text{Exp}(1)$.

Slick Way: Since the joint density of $(U,V)'$ is

$$f_{U,V}(u,v) = \begin{cases} v^{-2}e^{-u}, & \text{for } v \geq 1,\ u > 0, \\ 0, & \text{otherwise,} \end{cases}$$

we can immediately conclude that U and V are independent random variables with $f_V(v) = v^{-2}$ for $v \geq 1$ and $f_U(u) = e^{-u}$ for $u > 0$. And so we find (as before) that $U = X^2Y \in \text{Exp}(1)$.

Problem #24. Suppose that X and Y have joint density

$$f_{X,Y}(x,y) = \begin{cases} \lambda^2 e^{-\lambda y}, & \text{for } 0 < x < y, \\ 0, & \text{otherwise.} \end{cases}$$

Let $U = Y$ and $V = \frac{X}{Y-X}$ so that solving for X and Y gives

$$X = \frac{UV}{1+V} \quad \text{and} \quad Y = U.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v(1+v)^{-1} & u(1+v)^{-2} \\ 1 & 0 \end{vmatrix} = -\frac{u}{(1+v)^2}.$$

The density of $(U,V)'$ is therefore given by

$$f_{U,V}(u,v) = f_{X,Y}(uv(1+v)^{-1}, u) \cdot |J| = \lambda^2 e^{-\lambda u} \cdot \frac{u}{(1+v)^2} = \lambda^2 u\, e^{-\lambda u} \cdot \frac{1}{(1+v)^2},$$

provided that $0 < u < \infty$, $0 < v < \infty$. Since we can write the joint density as a product of a function of u only multiplied by a function of v only, we conclude that U and V are independent. That is,

$$f_{U,V}(u,v) = f_U(u) \cdot f_V(v)$$

where

$$f_U(u) = \lambda^2 u\, e^{-\lambda u} \text{ for } u > 0, \quad \text{and} \quad f_V(v) = \frac{1}{(1+v)^2} \text{ for } v > 0.$$

Notice that $U \in \Gamma(2, \lambda^{-1})$ and that $V \in F(2,2)$.