Independence of $\bar{X}$ and $S^2$ in a Normal Sample

The goal of this lecture is to prove that $\bar{X}$ and $S^2$ are independent for a normal sample; our proof of this theorem will follow Example 5.8.3.

**Theorem.** Suppose that $X_1, \ldots, X_n$ are independent $\mathcal{N}(0, 1)$ random variables. If

$$
\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \quad \text{and} \quad S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2
$$

 denote the sample mean and sample variance, respectively, then $\bar{X}$ and $S^2$ are independent.

**Proof.** Since $X_1, \ldots, X_n$ are i.i.d. $\mathcal{N}(0, 1)$, we conclude (using, say moment generating functions) that $\bar{X} \in \mathcal{N}(0, 1/n)$. Similarly, we can show that

$$
\bar{X} - X_j = \frac{1}{n} (X_1 + \cdots + X_{j-1} + X_{j+1} + \cdots + X_n) - \frac{n-1}{n} X_j \in \mathcal{N} \left( 0, \frac{n-1}{n^2} + \frac{(n-1)^2}{n^2} \right) = \mathcal{N} \left( 0, \frac{n-1}{n} \right)
$$

and so

$$
X_j - \bar{X} \in \mathcal{N} \left( 0, \frac{n-1}{n} \right)
$$

as well. We also note that

$$
\text{Cov}(X_j, \bar{X}) = \text{Cov} \left( X_j, \frac{1}{n} \sum_{i=1}^{n} X_i \right) = \frac{1}{n} \sum_{i=1}^{n} \text{Cov}(X_j, X_i) = \frac{1}{n} \text{Cov}(X_j, X_j) = \frac{1}{n},
$$

and so for $j \neq k$ it follows that

$$
\text{Cov}(X_j - \bar{X}, X_k - \bar{X}) = \text{Cov}(X_j, X_k) - \text{Cov}(X_j, \bar{X}) - \text{Cov}(X_k, \bar{X}) + \text{Cov}(\bar{X}, \bar{X})
$$

$$
= 0 - \frac{1}{n} - \frac{1}{n} + \frac{1}{n}
$$

$$
= -\frac{1}{n}
$$

using the fact that $\text{Cov}(\bar{X}, \bar{X}) = \text{Var}(\bar{X}) = 1/n$. Similarly,

$$
\text{Cov}(\bar{X}, X_j - \bar{X}) = \text{Cov}(X_j, \bar{X}) - \text{Cov}(\bar{X}, \bar{X})
$$

$$
= \frac{1}{n} - \frac{n}{n^2}
$$

$$
= 0.
$$

Thus, we see that $(\bar{X}, X_1 - \bar{X}, \ldots, X_n - \bar{X})' \in \mathcal{N}(\mathbf{0}, \mathbf{A})$ where

$$
\mathbf{A} = \begin{bmatrix}
1/n & 0 & 0 & \cdots & 0 \\
0 & (n-1)/n & -1/n & \cdots & -1/n \\
0 & -1/n & (n-1)/n & \cdots & -1/n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & -1/n & -1/n & \cdots & (n-1)/n
\end{bmatrix}
$$
By Theorem 5.7.2, we conclude from the form of $\mathbf{X}$ that $\mathbf{X}$ and $(X_1 - \overline{X}, \ldots, X_n - \overline{X})'$ are independent normal random vectors. It now follows from the transformation theorem (Theorem 1.2.1) that since $\mathbf{X}$ and $\mathbf{X} = (X_1 - \overline{X}, \ldots, X_n - \overline{X})'$ are independent, so too are $\mathbf{X}$ and $\mathbf{X}'\mathbf{X}$. Since

$$\mathbf{X}'\mathbf{X} = \sum_{i=1}^{n} (X_i - \overline{X})^2,$$

this implies that $\mathbf{X}$ and $S^2$ are independent.

Suppose that $Y_1, \ldots, Y_n$ are i.i.d. $\mathcal{N}(\mu, \sigma^2)$. We can use the previous result to show that

$$\overline{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i \quad \text{and} \quad S^2_Y = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \overline{Y})^2$$

are also independent. If we define $X_i = (Y_i - \mu)/\sigma$, then $X_i \in \mathcal{N}(0, 1)$. Therefore,

$$\overline{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{X_i - \mu}{\sigma} \right) = \frac{\overline{X} - \mu}{\sigma}$$

and

$$S^2_Y = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \overline{Y})^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left( \frac{X_i - \mu}{\sigma} - \frac{\overline{X} - \mu}{\sigma} \right)^2 = \frac{1}{\sigma^2} \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2 = \frac{S^2}{\sigma^2}.$$

Thus, since $\overline{X}$ and $S^2$ are independent, so too are

$$\frac{\overline{X} - \mu}{\sigma} = \overline{Y} \quad \text{and} \quad \frac{S^2}{\sigma^2} = S^2_Y.$$