Statistics 351–Intermediate Probability
Fall 2009 (200930)
Final Exam Solutions

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1. (a) We see that $f_{X,Y}(x,y) \geq 0$ for all x, $y \in \mathbb{R}$, and that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\int_0^y 15x^2y\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 5y^4\,\mathrm{d}y = y^5\Big|_0^1 = 1.$$

Thus, $f_{X,Y}$ is a legitimate density.

1. (b) We compute

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y = \int_x^1 15x^2y\,\mathrm{d}y = \frac{15}{2}x^2(1-x^2), \quad 0 < x < 1.$$

1. (c) We compute

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,\mathrm{d}x = \int_0^1 x \cdot \frac{15}{2}x^2(1-x^2)\,\mathrm{d}x = \frac{15}{2}\left[\frac{1}{4} - \frac{1}{6}\right] = \frac{5}{8}.$$

1. (d) We compute

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{15x^2y}{\frac{15}{2}x^2(1-x^2)} = \frac{2y}{1-x^2}, \quad x < y < 1.$$

1. (e) We compute

$$E(Y|X=x) = \int_{-\infty}^{\infty} y f_{Y|X=x}(y)\,\mathrm{d}y = \int_x^1 y \cdot \frac{2y}{1-x^2}\,\mathrm{d}y = \frac{2(1-x^3)}{3(1-x^2)}.$$

1. (f) Using properties of conditional expectation (Theorem 2.2.1), we compute

$$E(Y) = E(\,E(Y|X)\,) = E\left(\frac{2(1-X^3)}{3(1-X^2)}\right) = \int_0^1 \frac{2(1-x^3)}{3(1-x^2)} \cdot \frac{15}{2}x^2(1-x^2)\,\mathrm{d}x = 5\int_0^1 x^2 - x^5\,\mathrm{d}x = \frac{5}{6}.$$

1. (g) By definition, $\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$. We know that $E(X) = 5/8$ from **(c)** and $E(Y) = 5/6$ from **(f)**. Thus, we need to compute $E(XY)$. One way to do this is to use properties of conditional expectation (Theorem 2.2.1) and the result from **(e)** to write

$$E(XY) = E(E(XY|X)) = E(XE(Y|X)) = E\left(X \cdot \frac{2(1-X^3)}{3(1-X^2)}\right) = E\left(\frac{2X(1-X^3)}{3(1-X^2)}\right)$$

and proceed as in **(f)** to conclude

$$E\left(\frac{2X(1-X^3)}{3(1-X^2)}\right) = \int_0^1 \frac{2x(1-x^3)}{3(1-x^2)} \cdot \frac{15}{2}x^2(1-x^2)\,\mathrm{d}x = 5\int_0^1 x^3 - x^6\,\mathrm{d}x = \frac{15}{28}.$$

Alternatively, we can compute $E(XY)$ directly; that is,

$$E(XY) = \int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{\{0<x<y<1\}} xy \cdot 15x^2 y\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_0^y 15x^3y^2\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^1 y^2 \cdot \frac{15}{4}x^4 \Big|_{x=0}^{x=y} \mathrm{d}y = \frac{15}{4}\int_0^1 y^6\,\mathrm{d}y = \frac{15}{28}.$$

In either case, we conclude

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{15}{28} - \frac{5}{8}\cdot\frac{5}{6} = \frac{5}{336}.$$

2. (a) If $U = XY$ and $V = X$, then solving for X and Y gives $X = V$ and $Y = U/V$, so that the Jacobian of this transformation is

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1/v & -u/v^2 \end{vmatrix} = -\frac{1}{v}.$$

By Theorem 1.2.1, the joint density of $(U,V)'$ is therefore given by

$$f_{U,V}(u,v) = f_{X,Y}(v, u/v)\cdot|J| = 2(1-v)\cdot|-v^{-1}| = \frac{2(1-v)}{v} = \frac{2}{v} - 2$$

provided that $0 < u < v < 1$.

2. (b) It now follows that the density function of U is given by

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,\mathrm{d}v = \int_u^1 \frac{2}{v} - 2\,\mathrm{d}v = 2(\log|v| - v)\Big|_u^1 = 2(u - \log|u| - 1)$$

provided that $0 < u < 1$.

3. (a) If $X = 1/Z$, then the distribution function of X is

$$F_X(x) = P(X \le x) = P(1/Z \le x) = P(Z \ge 1/x) = 1 - P(Z \le x) = 1 - \int_{-\infty}^{1/x} \frac{b^a}{\Gamma(a)} z^{a-1}e^{-bz}\,\mathrm{d}z.$$

This implies that the density function of X is

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x}F_X(x) = -\frac{b^a}{\Gamma(a)}\frac{1}{x^{a-1}}e^{-b/x}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x} = \frac{b^a}{\Gamma(a)}\frac{1}{x^{a+1}}e^{-b/x}$$

provided $x > 0$.

3. (b) By definition,

$$f_{X,Y}(x,y) = f_{Y|X=x}(y)f_X(x) = \frac{1}{\sqrt{x\pi}}e^{-y^2/x}\cdot\frac{b^a}{\Gamma(a)}\frac{1}{x^{a+1}}e^{-b/x} = \frac{b^a}{\Gamma(a)\sqrt{\pi}}\frac{1}{x^{a+3/2}}\exp\left\{-\frac{1}{x}(y^2+b)\right\}$$

so that

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x = \int_0^{\infty} \frac{b^a}{\Gamma(a)\sqrt{\pi}}\frac{1}{x^{a+3/2}}\exp\left\{-\frac{1}{x}(y^2+b)\right\}\mathrm{d}x.$$

To perform this integration let $u = 1/x$, $\mathrm{d}u = -(1/x^2)\,\mathrm{d}x$ so that

$$f_Y(y) = \frac{b^a}{\Gamma(a)\sqrt{\pi}} \int_0^\infty u^{a-1/2} \exp\left\{-u(y^2+b)\right\} \mathrm{d}u.$$

Now let $v = u(y^2+b)$, $\mathrm{d}v = (y^2+b)\,\mathrm{d}u$ so that

$$\begin{aligned} f_Y(y) = \frac{b^a}{\Gamma(a)\sqrt{\pi}} \frac{1}{(y^2+b)^{a+1/2}} \int_0^\infty v^{a-1/2} e^{-v}\,\mathrm{d}v &= \frac{b^a}{\Gamma(a)\sqrt{\pi}} \frac{1}{(y^2+b)^{a+1/2}} \cdot \Gamma(a+1/2) \\ &= \frac{b^a\,\Gamma(a+1/2)}{\Gamma(a)\sqrt{\pi}\,(y^2+b)^{a+1/2}} \end{aligned}$$

provided $y > 0$. Note that Y has a generalized t distribution. That is, we can write the density of Y as

$$f_Y(y) = \frac{\Gamma(a+1/2)}{\sqrt{b\pi}\,\Gamma(a)} \frac{1}{(1+\frac{y^2}{b})^{a+1/2}},$$

and we see that if $a = n/2$ and $b = n$ for any positive integer n, then $Y \in t(n)$.

4. (a) Using properties of conditional expectation (Theorem 2.2.1), we compute

$$E(Y) = E(\,E(Y|X)\,) = E(2X) = 2E(X).$$

Since $E(Y) = 4$ and $E(X) = \alpha$, we conclude that $4 = 2\alpha$ or $\alpha = 2$.

4. (b) Using properties of conditional expectation (Corollary 2.2.3.1), we know

$$\mathrm{Var}(Y) = \mathrm{Var}(E(Y|X)) + E(\mathrm{Var}(Y|X))$$

and so

$$23 = \mathrm{Var}(2X) + E(X^2+\beta) = 4\,\mathrm{Var}(X) + E(X^2) + \beta = 4\cdot 2 + (2+2^2) + \beta = 14 + \beta.$$

Thus, $\beta = 9$.

5. The fact that X_1, X_2, X_3 are independent with $X_j \in \mathrm{Exp}(\lambda_j)$ so that $P(X_j > y) = e^{-y/\lambda_j}$ implies

$$\begin{aligned} P(Y > y) = P(\min\{X_1, X_2, X_3\} > y) &= P(X_1 > y, X_2 > y, X_3 > y) \\ &= P(X_1 > y)P(X_2 > y)P(X_3 > y) \\ &= e^{-y/\lambda_1} \cdot e^{-y/\lambda_2} \cdot e^{-y/\lambda_3} \\ &= \exp\left\{-y\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\right)\right\}. \end{aligned}$$

Thus, the distribution function of Y is

$$F_Y(y) = P(Y \le y) = 1 - P(Y > y) = 1 - \exp\left\{-y\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\right)\right\}$$

so that the density function of Y is

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\right) \exp\left\{-y\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\right)\right\}.$$

Note that $Y \in \mathrm{Exp}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\right)$.

6. (a) It follows from Theorem 4.3.1 that the joint density function of $(X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)})'$ is

$$f_{X_{(1)},X_{(2)},X_{(3)},X_{(4)}}(y_1, y_2, y_3, y_4) = 4!f(y_1)f(y_2)f(y_3)f(y_4) = \frac{24}{a^4}$$

provided that $0 < y_1 < y_2 < y_3 < y_4 < a$. Therefore, the joint density function of $(X_{(2)}, X_{(3)})'$ is

$$\begin{aligned} f_{X_{(2)},X_{(3)}}(y_2, y_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_{(1)},X_{(2)},X_{(3)},X_{(4)}}(y_1, y_2, y_3, y_4)\,\mathrm{d}y_3\,\mathrm{d}y_4 &= \int_0^{y_2}\int_{y_3}^{a} \frac{24}{a^4}\,\mathrm{d}y_4\,\mathrm{d}y_2 \\ &= \frac{24}{a^4} y_2(a - y_3) \end{aligned}$$

provided that $0 < y_2 < y_3 < a$.

6. (b) We see that

$$P(X_{(3)} < aX_{(2)}) = \iint_{\{y<ax\}} f_{X_{(2)},X_{(3)}}(x, y)\,\mathrm{d}x\,\mathrm{d}y = \frac{24}{a^4} \iint_{\left\{\substack{y<ax,\\ 0<x<y<a}\right\}} x(a - y)\,\mathrm{d}x\,\mathrm{d}y.$$

Draw the region of integration to see that it can be described as $\{y/a < x < y,\ 0 < y < a\}$ which implies

$$P(X_{(3)} < aX_{(2)}) = \frac{24}{a^4}\int_0^a \int_{y/a}^{y} x(a - y)\,\mathrm{d}x\,\mathrm{d}y.$$

We now find

$$\begin{aligned} \frac{24}{a^4}\int_0^a \int_{y/a}^{y} x(a - y)\,\mathrm{d}x\,\mathrm{d}y &= \frac{12}{a^4}\int_0^a (a-y)\left(y^2 - \frac{y^2}{a^2}\right)\mathrm{d}y = \frac{12}{a^4}\left(1 - \frac{1}{a^2}\right)\int_0^a y^2(a-y)\,\mathrm{d}y \\ &= \frac{12}{a^4}\left(1 - \frac{1}{a^2}\right)\left[\frac{ay^3}{3} - \frac{y^4}{4}\right]_0^a = \frac{12}{a^4}\left(1 - \frac{1}{a^2}\right)\cdot\frac{a^4}{12} = 1 - \frac{1}{a^2}. \end{aligned}$$

7. (a) Let

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that

$$B\mathbf{X} + \mathbf{b} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} X_1 + X_2 + X_3 \\ X_1 - X_2 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \mathbf{Y}.$$

By Theorem 5.3.1, we conclude that $\mathbf{Y} \in N(B\boldsymbol{\mu} + \mathbf{b}, B\boldsymbol{\Lambda}B')$ where

$$B\boldsymbol{\mu} + \mathbf{b} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$B\boldsymbol{\Lambda}B' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 3 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ 5 & -4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 9 \end{bmatrix}.$$

That is,

$$\mathbf{Y} \in N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 2 & 9 \end{bmatrix}\right).$$

7. (b) If we write $\Sigma = \text{Cov}(\mathbf{Y})$, then the density function of $\mathbf{Y}$ is

$$f_{\mathbf{Y}}(y_1, y_2) = \frac{1}{2\pi\sqrt{\det[\Sigma]}} \exp\left\{-\frac{1}{2}\mathbf{y}'\Sigma^{-1}\mathbf{y}\right\}.$$

Since $\det[\Sigma] = 36 - 4 = 32$ and

$$\Sigma^{-1} = \begin{bmatrix} 9/32 & -2/32 \\ -2/32 & 4/32 \end{bmatrix},$$

we conclude

$$\begin{aligned} f_{\mathbf{Y}}(y_1, y_2) &= \frac{1}{2\pi\sqrt{32}} \exp\left\{-\frac{1}{2}\left(\frac{9}{32}y_1^2 - \frac{4}{32}y_1y_2 + \frac{4}{32}y_2^2\right)\right\} \\ &= \frac{1}{2\pi\sqrt{32}} \exp\left\{-\frac{1}{2}\left(\frac{9}{32}y_1^2 - \frac{1}{8}y_1y_2 + \frac{1}{8}y_2^2\right)\right\}. \end{aligned}$$

7. (c) Since $Y_1 \in N(0, 4)$, we see that

$$f_{Y_1}(0) = \frac{1}{2\sqrt{2\pi}}.$$

Therefore,

$$f_{Y_2|Y_1=0}(y_2) = \frac{f_{\mathbf{Y}}(0, y_2)}{f_{Y_1}(0)} = \frac{\frac{1}{2\pi\sqrt{32}} \exp\left\{-\frac{1}{2}\left(\frac{1}{8}y_2^2\right)\right\}}{\frac{1}{2\sqrt{2\pi}}} = \frac{1}{\sqrt{2\pi}\sqrt{8}} \exp\left\{-\frac{1}{2 \cdot 8}y_2^2\right\}$$

which implies that $Y_2|Y_1 = 0 \in N(0, 8)$.

8. (a) In order to find the eigenvalues of $\mathbf{\Lambda}$, we must find those values of λ such that $\det[\mathbf{\Lambda} - \lambda I] = 0$. Therefore,

$$\det[\mathbf{\Lambda} - \lambda I] = \det\begin{bmatrix} \frac{7}{4} - \lambda & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{5}{4} - \lambda \end{bmatrix} = \left(\frac{7}{4} - \lambda\right)\left(\frac{5}{4} - \lambda\right) - \frac{3}{16} = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$$

so that the eigenvalues of $\mathbf{\Lambda}$ are $\lambda_1 = 1$ and $\lambda_2 = 2$.

8. (b) Since $\lambda_1 = 1$,

$$[\mathbf{\Lambda} - \lambda_1 I \,|\, \mathbf{0}] = \left[\begin{array}{cc|c} \frac{3}{4} & -\frac{\sqrt{3}}{4} & 0 \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & -\sqrt{3} & 0 \\ -\sqrt{3} & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -\sqrt{3} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

and since $\lambda_2 = 2$,

$$[\mathbf{\Lambda} - \lambda_2 I \,|\, \mathbf{0}] = \left[\begin{array}{cc|c} -\frac{1}{4} & -\frac{\sqrt{3}}{4} & 0 \\ -\frac{\sqrt{3}}{4} & -\frac{3}{4} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & -\sqrt{3} & 0 \\ -\sqrt{3} & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \sqrt{3} & 0 \\ 0 & 0 & 0 \end{array}\right],$$

we conclude that eigenvectors for λ_1 and λ_2 are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix},$$

respectively. Therefore, the diagonal matrix is

$$D = \text{diag}(\lambda_1, \lambda_2) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

and the orthogonal matrix is

$$C = \begin{bmatrix} \frac{\mathbf{v}_1}{||\mathbf{v}_1||} & \frac{\mathbf{v}_2}{||\mathbf{v}_2||} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

since $||\mathbf{v}_1|| = ||\mathbf{v}_2|| = 2$.

8. (c) If $\mathbf{Y} = C'\mathbf{X}$, then by Theorem 5.3.1, $\mathbf{Y}$ is MVN with mean $C'\boldsymbol{\mu}$ and covariance matrix

$$C'\mathbf{\Lambda}C'' = C'\mathbf{\Lambda}C = C'(CDC')C = (C'C)D(CC') = IDI = D$$

using our result from **(b)**. Hence, we conclude

$$\mathbf{Y} \in N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right).$$

8. (d) Since $\mathbf{Y}$ has a multivariate normal distribution, we know from Theorem 5.7.1 that the components of $\mathbf{Y}$ are independent if and only if they are uncorrelated. From **(c)** we know that $\text{Cov}(Y_1, Y_2) = 0$ so that Y_1 and Y_2 are, in fact, independent.

9. (a) It follows from Theorem 5.3.1 that $\mathbf{Y} = (Y_1, Y_2)'$ has a multivariate normal distribution since the components of $\mathbf{Y}$ are linear combinations of the components of $\mathbf{X} \in N(\boldsymbol{\mu}, \mathbf{\Lambda})$. Therefore, Y_1 and Y_2 are each normally distributed. Hence, we simply need to compute the mean and variance of Y_1 and Y_2. Since $Y_2 = X_2$, and $X_2 \in N(0,1)$, we immediately conclude $Y_2 \in N(0,1)$. As for Y_1, we compute

$$E(Y_1) = \frac{E(X_1) - \rho E(X_2)}{\sqrt{1-\rho^2}} = 0$$

and

$$\begin{aligned} \text{Var}(Y_1) = \text{Var}\left(\frac{X_1 - \rho X_2}{\sqrt{1-\rho^2}}\right) &= \frac{\text{Var}(X_1 - \rho X_2)}{1-\rho^2} \\ &= \frac{\text{Cov}(X_1 - \rho X_2, X_1 - \rho X_2)}{1-\rho^2} \\ &= \frac{\text{Cov}(X_1, X_1) - 2\rho\,\text{Cov}(X_1, X_2) + \rho^2\,\text{Cov}(X_2, X_2)}{1-\rho^2} \\ &= \frac{1 - 2\rho^2 + \rho^2}{1-\rho^2} \\ &= 1 \end{aligned}$$

since $X_1 \in N(0,1)$ and $X_2 \in N(0,1)$ with $\text{Cov}(X_1, X_2) = \rho$. Finally, we know from Theorem 5.7.1 that Y_1 and Y_2 are independent if and only if $\text{Cov}(Y_1, Y_2) = 0$. Since

$$\text{Cov}(Y_1, Y_2) = \text{Cov}\left(\frac{X_1 - \rho X_2}{\sqrt{1-\rho^2}}, X_2\right) = \frac{\text{Cov}(X_1, X_2) - \rho\,\text{Cov}(X_2, X_2)}{\sqrt{1-\rho^2}} = \frac{\rho - \rho}{\sqrt{1-\rho^2}} = 0,$$

we conclude that Y_1 and Y_2 are, in fact, independent $N(0,1)$ random variables.

9. (b) Since Y_1 and Y_2 are independent $N(0,1)$ random variables, we know that Y_1^2 and Y_2^2 are independent $\chi^2(1)$ random variables. Therefore, $Y_1^2 + Y_2^2 \in \chi^2(2)$ and so

$$\begin{aligned} Y_1^2 + Y_2^2 = \left(\frac{X_1 - \rho X_2}{\sqrt{1-\rho^2}}\right)^2 + X_2^2 &= \frac{X_1^2 - 2\rho X_1 X_2 + \rho^2 X_2^2}{1-\rho^2} + X_2^2 \\ &= \frac{X_1^2 - 2\rho X_1 X_2 + \rho^2 X_2^2 + (1-\rho^2)X_2^2}{1-\rho^2} \\ &= \frac{X_1^2 - 2\rho X_1 X_2 + X_2^2}{1-\rho^2} \in \chi^2(2). \end{aligned}$$

10. We begin by noting that $E(Z_1) = E(X_1) + E(Y_1) = 0$ and $E(Z_2) = E(X_2) + E(Y_2) = 0$ so that

$$E(\mathbf{Z}) = \begin{bmatrix} E(Z_1) \\ E(Z_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $\mathbf{X}$ and $\mathbf{Y}$ are independent, we conclude that

$$\operatorname{Cov}(X_1, Y_1) = \operatorname{Cov}(X_1, Y_2) = \operatorname{Cov}(X_2, Y_1) = \operatorname{Cov}(X_2, Y_2) = 0.$$

Hence, we compute

$$\operatorname{Var}(Z_1) = \operatorname{Var}(X_1 + Y_1) = \operatorname{Var}(X_1) + \operatorname{Var}(Y_1) + 2\operatorname{Cov}(X_1, Y_1) = 1 + 1 + 0 = 2,$$

$$\operatorname{Var}(Z_2) = \operatorname{Var}(X_2 + Y_2) = \operatorname{Var}(X_2) + \operatorname{Var}(Y_2) + 2\operatorname{Cov}(X_2, Y_2) = 1 + 1 + 0 = 2$$

and

$$\begin{aligned} \operatorname{Cov}(Z_1, Z_2) = \operatorname{Cov}(X_1 + Y_1, X_2 + Y_2) &= \operatorname{Cov}(X_1, X_2) + \operatorname{Cov}(X_1, Y_2) + \operatorname{Cov}(X_2, Y_1) + \operatorname{Cov}(X_2, Y_2) \\ &= \rho + 0 + 0 - \rho \\ &= 0 \end{aligned}$$

so that

$$\operatorname{Cov}(\mathbf{Z}) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

At this point, we see that $\mathbf{Z}$ has the required mean vector and covariance matrix. The final step is to prove that $\mathbf{Z}$ has a multivariate normal distribution. By Definition I it is sufficient to show that $a_1 Z_1 + a_2 Z_2$ has a one-dimensional normal distribution for any a_1, $a_2 \in \mathbb{R}$. Since $\mathbf{X}$ is multivariate normal, we know from Definition I that $a_1 X_1 + a_2 X_2$ has a one-dimensional normal distribution, and we also know from Definition I that $a_1 Y_1 + a_2 Y_2$ has a one-dimensional normal distribution. Since $\mathbf{X}$ and $\mathbf{Y}$ are independent, we know that $a_1 X_1 + a_2 X_2$ and $a_1 Y_1 + a_2 Y_2$ are necessarily independent. Since the sum of independent one-dimensional normal distributions has a normal distribution, we know that

$$(a_1 X_1 + a_2 X_2) + (a_1 Y_1 + a_2 Y_2) = a_1(X_1 + Y_1) + a_2(X_2 + Y_2) = a_1 Z_1 + a_2 Z_2$$

has a one-dimensional normal distribution. Thus,

$$\mathbf{Z} = (Z_1, Z_2)' \in N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right)$$

as required.

11. (a) Since

$$f(x) = \frac{\sqrt{2}}{\sqrt{\pi}}\, e^{-x^2/2}, \quad x > 0,$$

we conclude that

$$F(x) = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^x e^{-u^2/2}\, \mathrm{d}u$$

for $x > 0$ (and $F(x) = 0$ for $x \le 0$). Using Theorem 4.1.2, the density function of Y, the minimum of $n = 2$ i.i.d. random variables with common density function $f(x)$ and common distribution function $F(x)$ is

$$\begin{aligned} f_Y(y) = 2f(y)[1 - F(y)] &= 2 \cdot \frac{\sqrt{2}}{\sqrt{\pi}}\, e^{-y^2/2} \cdot \left[1 - \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^y e^{-u^2/2}\, \mathrm{d}u\right] \\ &= 2 \cdot \frac{\sqrt{2}}{\sqrt{\pi}}\, e^{-y^2/2} \cdot \frac{\sqrt{2}}{\sqrt{\pi}} \int_y^\infty e^{-u^2/2}\, \mathrm{d}u \\ &= \frac{4}{\pi}\, e^{-y^2/2} \int_y^\infty e^{-u^2/2}\, \mathrm{d}u \end{aligned}$$

provided that $y > 0$.

11. (b) We find

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f_Y(y)\, \mathrm{d}y = \int_0^\infty y^2 \left[\frac{4}{\pi}\, e^{-y^2/2} \int_y^\infty e^{-u^2/2}\, \mathrm{d}u\right] \mathrm{d}y = \frac{4}{\pi} \int_0^\infty \int_y^\infty y^2 e^{-y^2/2} e^{-u^2/2}\, \mathrm{d}u\, \mathrm{d}y.$$

11. (c) In order to evaluate the double integral from **(b)** we switch to polar coordinates. That is, let $u = r\cos\theta$, $y = r\sin\theta$, $\mathrm{d}u\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$ so that the region $\{0 \le y < \infty,\, y \le u < \infty\}$ in cartesian coordinates corresponds to $\{0 \le r < \infty,\, 0 \le \theta < \pi/4\}$ in polar coordinates. Thus, we conclude

$$\begin{aligned} \frac{4}{\pi} \int_0^\infty \int_y^\infty y^2 e^{-y^2/2} e^{-u^2/2}\, \mathrm{d}u\, \mathrm{d}y &= \frac{4}{\pi} \int_0^{\pi/4} \int_0^\infty (r\sin\theta)^2\, e^{-r^2/2} \cdot r\, \mathrm{d}r\, \mathrm{d}\theta \\ &= \frac{4}{\pi} \int_0^{\pi/4} \sin^2\theta\, \mathrm{d}\theta \int_0^\infty r^3 e^{-r^2/2}\, \mathrm{d}r \end{aligned}$$

To evaluate this make the substitution $v = r^2/2$, $\mathrm{d}v = r\,\mathrm{d}r$ so that

$$\int_0^\infty r^3 e^{-r^2/2}\, \mathrm{d}r = \int_0^\infty 2ve^{-v}\, \mathrm{d}v = 2\int_0^\infty v^{2-1} e^{-v}\, \mathrm{d}v = 2\Gamma(2) = 2,$$

and use the identity $\cos(2\theta) = 1 - 2\sin^2(\theta)$ to write

$$\int_0^{\pi/4} \sin^2\theta\, \mathrm{d}\theta = \int_0^{\pi/4} \frac{1 - \cos(2\theta)}{2}\, \mathrm{d}\theta = \left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right]_0^{\pi/4} = \frac{\pi}{8} - \frac{1}{4}.$$

Therefore, we conclude that

$$E(Y^2) = \frac{4}{\pi} \cdot 2 \cdot \left[\frac{\pi}{8} - \frac{1}{4}\right] = 1 - \frac{2}{\pi}.$$

Note that this is a special case of *Youden's Angel Problem* which was originally posed in 1953 by W.J. Youden. For further discussion, see *Two problems in sets of measurements* by M.G. Kendall (*Biometrika*, Vol. 41, No. 3/4 (Dec. 1954), pp. 560–564).