Stat 351 Fall 2009
Assignment #6 Solutions

1. Recall from Stat 251 that if $X \in N(0,1)$, then $X^2 \in \chi^2(1)$. Furthermore, recall that if $Z_1, \dots, Z_n$ are independent with $Z_j \in \chi^2(p_j)$, then

$$\sum_{j=1}^{n} Z_j \in \chi^2(p_1 + \cdots + p_n).$$

(That is, the sum of independent chi-squared random variables is itself chi-squared with degrees of freedom additive.) Since

$$\mathbf{X}'\mathbf{X} = X_1^2 + X_2^2 + \cdots + X_n^2$$

is the sum of n i.i.d. $\chi^2(1)$ random variables, we conclude

$$\mathbf{X}'\mathbf{X} \in \chi^2(n).$$

2. (a) Since X and Y are i.i.d. $N(0,1)$, we know that

$$3X + 4Y \in N(0, 3^2 + 4^2) = N(0, 25).$$

Normalizing implies that

$$Z = \frac{3X + 4Y}{5} \in N(0,1).$$

Thus,

$$P(3X + 4Y > 5) = P(Z > 1) \doteq 0.1587$$

using a table of normal probabilities.

2. (b) Since X and Y are independent, we know that

$$P(\min\{X,Y\} > 1) = P(X > 1, Y > 1) = P(X > 1) \cdot P(Y > 1) \doteq (0.1587)^2$$

and so

$$P(\min\{X,Y\} < 1) \doteq 1 - (0.1587)^2 \doteq 0.9748$$

using a table of normal probabilities.

2. (c) Since

$$P(|\min\{X,Y\}| < 1) = P(-1 < \min\{X,Y\} < 1) = P(\min\{X,Y\} < 1) - P(\min\{X,Y\} < -1)$$

and

$$P(\min\{X,Y\} < -1) = 1 - P(\min\{X,Y\} > -1) = 1 - P(X > -1) \cdot P(Y > -1) \doteq 1 - (0.8413)^2$$

we conclude that

$$\begin{aligned} P(|\min\{X,Y\}| < 1) &\doteq [1 - (0.1587)^2] - [1 - (0.8413)^2] = (0.8413)^2 - (0.1587)^2 \\ &\doteq 0.6826 \end{aligned}$$

using a table of normal probabilities.

2. (d) Notice that

$$\max\{X,Y\} - \min\{X,Y\} = |X - Y|$$

and that $X - Y \in N(0,2)$. Normalizing implies

$$Z = \frac{X - Y}{\sqrt{2}} \in N(0,1)$$

and so we find

$$\begin{aligned} P(\max\{X,Y\} - \min\{X,Y\} < 1) = P(|X - Y| < 1) &= P(|Z| < 1/\sqrt{2}\,) \\ &= P(-1/\sqrt{2} < Z < 1/\sqrt{2}\,) \\ &\doteq 0.5205 \end{aligned}$$

using a table of normal probabilities.

2. (e) Note that $X^2 + Y^2 \in \chi^2(2)$ as in Problem **1**. However, we know that $\chi^2(2) = \Gamma(1,2) = \text{Exp}(2)$. Thus, if $Z = X^2 + Y^2$ so that $Z \in \text{Exp}(2)$, then

$$P(X^2 + Y^2 \le 1) = P(Z \le 1) = 1 - e^{-1/2}.$$

3. (a) By Definition I, we see that $X_1 - \rho X_2$ is normally distributed with mean

$$E(X_1 - \rho X_2) = E(X_1) - \rho E(X_2) = 0$$

and variance

$$\text{var}(X_1 - \rho X_2) = \text{var}(X_1) + \rho^2\,\text{var}(X_2) - 2\rho\,\text{cov}(X_1, X_2) = 1 + \rho^2 - 2\rho^2 = 1 - \rho^2.$$

That is, $X_1 - \rho X_2 = Y$ where $Y \in N(0, 1 - \rho^2)$. Hence, $Y = \sqrt{1-\rho^2}Z$ where $Z \in N(0,1)$. In other words, there exists a $Z \in N(0,1)$ such that

$$X_1 - \rho X_2 = \sqrt{1-\rho^2}Z.$$

3. (b) Since $\mathbf{X} = (X_1, X_2)'$ is MVN, and since

$$Z = \frac{X_1}{\sqrt{1-\rho^2}} - \frac{\rho X_2}{\sqrt{1-\rho^2}},$$

we conclude that $(Z, X_2)'$ is also a MVN. Hence, we know from Theorem 5.7.1 that the components of a MVN are independent if and only if they are uncorrelated. We find

$$\begin{aligned} \text{cov}(Z, X_2) = \text{cov}\left(\frac{X_1}{\sqrt{1-\rho^2}} - \frac{\rho X_2}{\sqrt{1-\rho^2}}, X_2\right) &= \frac{1}{\sqrt{1-\rho^2}}\,\text{cov}(X_1, X_2) - \frac{\rho}{\sqrt{1-\rho^2}}\,\text{var}(X_2) \\ &= \frac{\rho}{\sqrt{1-\rho^2}} - \frac{\rho}{\sqrt{1-\rho^2}} \\ &= 0 \end{aligned}$$

which verifies that Z and X_2 are, in fact, independent.

Exercise 5.3, page 126. Let

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

so that $\mathbf{Y} = B\mathbf{X}$. By Theorem 5.3.1, $\mathbf{Y}$ is MVN with mean

$$B\bar{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and covariance matrix

$$B\mathbf{\Lambda}B' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 7/2 & 1/2 & -1 \\ 1/2 & 1/2 & 0 \\ -1 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 5 \end{pmatrix}.$$

Hence, we see that $\mathbf{Y} \in N(\bar{0}, \mathbf{\Sigma})$ where

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 5 \end{pmatrix}.$$

We now compute $\det[\mathbf{\Sigma}] = 10 - 9 = 1$ and

$$\mathbf{\Sigma}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & -3 \\ 0 & -3 & 2 \end{pmatrix}.$$

If we write $\mathbf{y} = (y_1, y_2, y_3)'$, then

$$\mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{y} = y_1^2 + 5y_2^2 - 6y_2y_3 + 2y_3^2$$

and so the density of $\mathbf{Y}$ is given by

$$f_{\mathbf{Y}}(\mathbf{y}) = \left(\frac{1}{2\pi}\right)^{3/2} \exp\left\{-\frac{1}{2}(y_1^2 + 5y_2^2 - 6y_2y_3 + 2y_3^2)\right\}.$$

Note that this problem could also be solved by observing that $Y_1 \in N(0,1)$ and

$$(Y_2, Y_3)' \in N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}\right)$$

are independent so that $f_{\mathbf{Y}}(\mathbf{y}) = f_{Y_1}(y_1) \cdot f_{Y_2,Y_3}(y_2, y_3)$.