Stat 351 Fall 2009
Assignment #3 Solutions

Problem 1: We find

$$P(X < Y < Z) = \iiint\limits_{\{x<y<z\}} f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Since $f(x, y, z) = e^{-x-y-z}$ for $0 < x, y, z < \infty$, the six equivalent iterated integrals for this expression are

$$\int_0^\infty \int_0^z \int_0^y e^{-x-y-z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^\infty \int_y^\infty \int_0^y e^{-x-y-z}\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y = \int_0^\infty \int_0^z \int_x^z e^{-x-y-z}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z$$

$$= \int_0^\infty \int_x^\infty \int_x^z e^{-x-y-z}\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x = \int_0^\infty \int_0^y \int_y^\infty e^{-x-y-z}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y = \int_0^\infty \int_x^\infty \int_y^\infty e^{-x-y-z}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

All of them integrate to 1/6; for example, the first of these iterated integrals is

$$\begin{aligned}
\int_0^\infty \int_0^z \int_0^y e^{-x-y-z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_0^\infty \int_0^z e^{-y-z} \int_0^y e^{-x}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_0^\infty \int_0^z e^{-y-z}(1 - e^{-y})\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_0^\infty e^{-z} \int_0^z (e^{-y} - e^{-2y})\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_0^\infty e^{-z} \left[(1 - e^{-z}) - \left(\frac{1}{2} - \frac{1}{2}e^{-2z}\right)\right] \mathrm{d}z \\
&= \int_0^\infty \left(\frac{1}{2}e^{-z} - e^{-2z} + \frac{1}{2}e^{-3z}\right) \mathrm{d}z \\
&= \frac{1}{2} - \frac{1}{2} + \frac{1}{6}
\end{aligned}$$

as expected.