Stat 351 Fall 2009
Assignment #2

This assignment is due at the beginning of class on Friday, October 23, 2009.

1. The purpose of this exercise is to lead you through the verification that the density function of a $\beta(a,b)$ random variable, namely

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, \quad 0 < x < 1,$$

is, in fact, a legitimate density. Suppose that

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,\mathrm{d}x. \qquad (*)$$

Our goal is to show that

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

(a) As a first step, we will need to prove that

$$B(a,b) = 2\int_0^{\pi/2} \cos^{2a-1}(\theta)\sin^{2b-1}(\theta)\,\mathrm{d}\theta.$$

This is done by making the substitution $x = \cos^2(\theta)$ in $(*)$.

(b) Now consider

$$\Gamma(a)\Gamma(b) = \int_0^\infty u^{a-1}e^{-u}\,\mathrm{d}u \cdot \int_0^\infty v^{b-1}e^{-v}\,\mathrm{d}v = \int_0^\infty\int_0^\infty u^{a-1}v^{b-1}e^{-(u+v)}\,\mathrm{d}u\,\mathrm{d}v.$$

Change variables by letting $u = x^2$ and $v = y^2$ to show that

$$\Gamma(a)\Gamma(b) = 4\int_0^\infty\int_0^\infty x^{2a-1}y^{2b-1}e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y.$$

(c) Change **(b)** to polar coordinates with $x = r\cos(\theta)$, $y = r\sin(\theta)$ for $0 \le r < \infty$, $0 \le \theta \le \pi/2$ to show that

$$\Gamma(a)\Gamma(b) = 4\int_0^\infty r^{2a+2b-2}e^{-r^2}r\,\mathrm{d}r \int_0^{\pi/2} \cos^{2a-1}(\theta)\sin^{2b-1}(\theta)\,\mathrm{d}\theta. \qquad (**)$$

(d) Let $t = r^2$ to show that

$$2\int_0^\infty r^{2a+2b-2}e^{-r^2}r\,\mathrm{d}r = \int_0^\infty t^{a+b-1}e^{-t}\,\mathrm{d}t = \Gamma(a+b).$$

(e) Combine **(a)** and **(d)** to conclude from $(**)$ that

$$\Gamma(a)\Gamma(b) = \Gamma(a+b)B(a,b)$$

as required.

2. Chapter 2 Problems, pages 50–55, #2, #8 through #11, #18, #19, #22, #23, #30