Statistics 351 (Fall 2008)
The Gamma Function

Suppose that $p > 0$, and define

$$\Gamma(p) := \int_0^\infty u^{p-1} e^{-u} \, du.$$

We call $\Gamma(p)$ the *Gamma function* and it appears in many of the formulæ of density functions for continuous random variables such as the Gamma distribution, Beta distribution, Chi-squared distribution, t distribution, and F distribution.

The first thing that should be checked is that the integral defining $\Gamma(p)$ is convergent for $p > 0$. For now, we will assume that it is true that the Gamma function is well-defined. This will allow us to derive some of its important properties and show its utility for statistics.

The Gamma function may be viewed as a generalization of the factorial function as this first result shows.

Proposition 1. *If $p > 0$, then $\Gamma(p+1) = p\,\Gamma(p)$.*

Proof. This is proved using integration by parts from first-year calculus. Indeed,

$$\Gamma(p+1) = \int_0^\infty u^{p+1-1} e^{-u} \, du = \int_0^\infty u^p e^{-u} \, du = -u^p e^{-u} \Big|_0^\infty + \int_0^\infty p u^{p-1} e^{-u} \, du = 0 + p\,\Gamma(p).$$

To do the integration by parts, let $w = u^p$, $dw = pu^{p-1}$, $dv = e^{-u}$, $v = -e^{-u}$ and recall that $\int w \, dv = wv - \int v \, dw$. $\square$

If p is an integer, then we have the following corollary.

Corollary 2. *If n is a positive integer, then $\Gamma(n) = (n-1)!$.*

Proof. Using the previous proposition, we see that

$$\Gamma(n) = (n-1)\,\Gamma(n-1) = (n-1)(n-2)\,\Gamma(n-2) = \cdots = (n-1)(n-2)\cdots 2 \cdot \Gamma(1).$$

However,

$$\Gamma(1) = \int_0^\infty u^0 e^{-u} \, du = \int_0^\infty e^{-u} \, du = -e^{-u} \Big|_0^\infty = 1 \tag{1}$$

and so

$$\Gamma(n) = (n-1)(n-2)\cdots 2 \cdot 1 = (n-1)!$$

as required. $\square$

The next proposition shows us how to calculate $\Gamma(p)$ for certain fractional values of p.

Proposition 3. $\Gamma(1/2) = \sqrt{\pi}$.

Proof. By definition,

$$\Gamma(1/2) = \int_0^\infty u^{-1/2} e^{-u}\, du.$$

Making the substitution $u = v^2$ so that $du = 2v\, dv$ gives

$$\int_0^\infty u^{-1/2} e^{-u}\, du = \int_0^\infty v^{-1} e^{-v^2}\, 2v\, dv = 2\int_0^\infty e^{-v^2}\, dv = \int_{-\infty}^\infty e^{-v^2}\, dv$$

where the last equality follows since e^{-v^2} is an even function. We now recognize this as the density function of a $\mathcal{N}(0, 1/2)$ random variable. That is,

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{v^2}{2\sigma^2}}\, dv = 1$$

and so

$$\int_{-\infty}^\infty e^{-\frac{v^2}{2\sigma^2}}\, dv = \sigma\sqrt{2\pi}.$$

Choosing $\sigma^2 = 1/2$ gives

$$\int_{-\infty}^\infty e^{-v^2}\, dv = \sqrt{\pi}$$

and so we conclude that $\Gamma(1/2) = \sqrt{\pi}$ as claimed. □

This proposition can be combined with Proposition 1 to show, for example, that

$$\Gamma(3/2) = \Gamma(1/2 + 1) = 1/2 \cdot \Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

and

$$\Gamma(5/2) = \Gamma(3/2 + 1) = 3/2 \cdot \Gamma(3/2) = \frac{3\sqrt{\pi}}{4}.$$

For students, though, perhaps the most powerful use of the Gamma function is to compute integrals such as the following.

Example 4. Suppose that $Y \sim \text{Exp}(\theta)$. Use Gamma functions to quickly compute $\mathbb{E}(Y^2)$.

Solution. By definition, we have

$$\mathbb{E}(Y^2) = \int_{-\infty}^\infty y^2 f_Y(y)\, dy = \frac{1}{\theta}\int_0^\infty y^2 e^{-y/\theta}\, dy.$$

Make the substitution $u = y/\theta$ so that $dy = \theta du$. This gives

$$\frac{1}{\theta}\int_0^\infty y^2 e^{-y/\theta}\, dy = \frac{1}{\theta}\int_0^\infty \theta^2 u^2 e^{-u}\, \theta\, du = \theta^2 \int_0^\infty u^2 e^{-u}\, du = \theta^2\, \Gamma(3).$$

By Corollary 2, $\Gamma(3) = (3-1)! = 2$ and so $\mathbb{E}(Y^2) = 2\,\theta^2$.

Example 5. If $Y \sim \text{Exp}(\theta)$, then this method can be applied to compute $\mathbb{E}(Y^k)$ for any positive integer k. Indeed,

$$\mathbb{E}(Y^k) = \frac{1}{\theta}\int_0^\infty y^k e^{-y/\theta}\, dy = \frac{1}{\theta}\int_0^\infty \theta^k u^k e^{-u}\,\theta\, du = \theta^k\,\Gamma(k+1) = k!\,\theta^k.$$

Theorem 6. *For $p > 0$, the integral*

$$\int_0^\infty u^{p-1}\, e^{-u}\, du$$

is absolutely convergent.

Proof. Since we are considering the value of the improper integral

$$\int_0^\infty u^{p-1}\, e^{-u}\, du$$

for all $p > 0$, there is need to be careful at both endpoints 0 and ∞.

We begin with the easiest case. If $p = 1$, then

$$\int_0^\infty u^0 e^{-u}\, du = \int_0^\infty e^{-u}\, du = \lim_{N\to\infty}\int_0^N e^{-u}\, du = \lim_{N\to\infty}(1 - e^{-N}) = 1.$$

For the remaining cases $0 < p < 1$ and $p > 1$ we will consider the integral from 0 to 1 and the integral from 1 to ∞ separately.

If $0 < p < 1$, then the integral

$$\int_0^1 u^{p-1}\, e^{-u}\, du$$

is improper. Thus,

$$\int_0^1 u^{p-1}\, e^{-u}\, du = \lim_{a\to 0+}\int_a^1 u^{p-1}\, e^{-u}\, du \le \lim_{a\to 0+}\int_a^1 u^{p-1}\, du = \lim_{a\to 0+}\frac{1-a^p}{p} = \frac{1}{p}$$

since $e^{-u} \le 1$ for $0 \le u \le 1$.

Furthermore, if $0 < p < 1$, then $0 < u^{p-1} \le 1$ for $u \ge 1$ and so

$$\int_1^\infty u^{p-1}\, e^{-u}\, du = \lim_{N\to\infty}\int_1^N u^{p-1}\, e^{-u}\, du \le \lim_{N\to\infty}\int_1^N e^{-u}\, du = \lim_{N\to\infty}(1 - e^{-N}) = 1.$$

Thus, we can conclude that for $0 < p < 1$,

$$\int_0^\infty u^{p-1}\, e^{-u}\, du = \int_0^1 u^{p-1}\, e^{-u}\, du + \int_1^\infty u^{p-1}\, e^{-u}\, du \le \frac{1}{p} + 1 < \infty.$$

If $p > 1$, then $u^{p-1} \in [0,1]$ and $e^{-u} \leq 1$ for $0 \leq u \leq 1$. Thus,

$$\int_0^1 u^{p-1}\, e^{-u}\, du \leq \int_0^1 u^{p-1}\, du = \frac{u^p}{p}\bigg|_0^1 = \frac{1}{p}.$$

On the other hand, if $p > 1$, then notice that $p - \lfloor p \rfloor \in [0,1)$ so that $0 < u^{p-\lfloor p \rfloor - 1} \leq 1$ for $u \geq 1$. We then have

$$\int_1^N u^{p-1}\, e^{-u}\, du = \int_1^N u^{p-\lfloor p \rfloor -1} u^{\lfloor p \rfloor}\, e^{-u}\, du \leq \int_1^N u^{\lfloor p \rfloor}\, e^{-u}\, du.$$

Thus, integration by parts $\lfloor p \rfloor$ times (the so-called *reduction formula*) gives

$$\begin{aligned}
&\int_1^N u^{\lfloor p \rfloor}\, e^{-u}\, du \\
&\qquad = -e^{-u}\left(u^{\lfloor p \rfloor} + \lfloor p \rfloor u^{\lfloor p \rfloor - 1} + \lfloor p \rfloor \cdot (\lfloor p \rfloor - 1) u^{\lfloor p \rfloor - 2} + \cdots + \lfloor p \rfloor \cdot (\lfloor p \rfloor - 1) \cdots 2 \cdot u\right)\bigg|_1^N \\
&\qquad\qquad + \lfloor p \rfloor \cdot (\lfloor p \rfloor - 1) \cdots 2 \cdot 1 \cdot \int_1^N e^{-u}\, du
\end{aligned}$$

and so

$$\lim_{N \to \infty} \int_1^N u^{\lfloor p \rfloor}\, e^{-u}\, du = \lfloor p \rfloor\,!.$$

Thus, we can conclude that for $p > 1$,

$$\int_0^\infty u^{p-1}\, e^{-u}\, du = \int_0^1 u^{p-1}\, e^{-u}\, du + \int_1^\infty u^{p-1}\, e^{-u}\, du \leq \frac{1}{p} + \lfloor p \rfloor\,! < \infty.$$

In every case we have $u^{p-1}\, e^{-u} \geq 0$ and so

$$\int_0^\infty \left|u^{p-1}\, e^{-u}\right|\, du = \int_0^\infty u^{p-1}\, e^{-u}\, du < \infty.$$

That is, this integral is absolutely convergent, and so $\Gamma(p)$ is well-defined for $p > 0$. $\square$